The Essence of Generalized Algebraic Data Types

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Type-indexed vectors

```
data Zero :: *
data Succ :: * -> *

data Vec :: * -> * -> * where
  Nil :: forall a. Vec a Zero
  Cons :: forall a n. a -> Vec a n -> Vec a (Succ n)
```

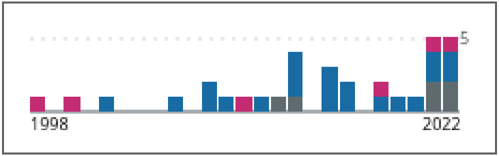
Fixed-length lists without dependent types.

Well-typed DSLs

```
data Expr :: * -> * where
  LiftExpr :: forall a. a -> Expr a
  LamExpr :: forall a b. (a -> Expr b) -> Expr (a -> b)
  AppExpr :: forall a b. Expr (a -> b)
                         -> Expr a -> Expr b
  FixExpr :: forall a. Expr (a -> a) -> Expr a
  ...
```

Can be used to embed DSLs, and fallback onto Haskell for typechecking.
E.g., Pugs, Darcs.

Existing works on GADTs



31 relevant results on dblp:

- 8: categorical semantics
- 23: syntax, type-inference, implementation, usage

- Our work can be used by compiler developers, language designers, or for the verification of data structures.
- Moreover, we target feature-rich languages.

This work

- Calculus for GADTs.
- Semantic model for this calculus.

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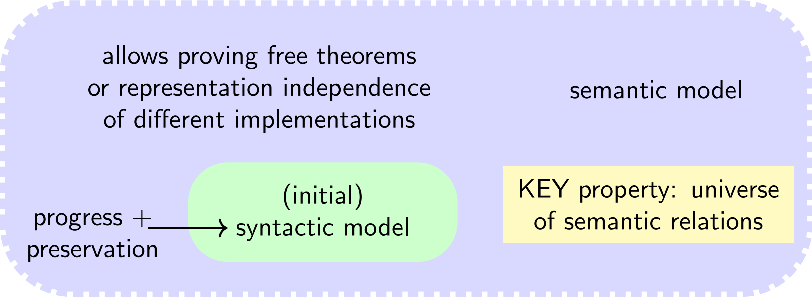

allows proving free theorems
or representation independence
of different implementations

semantic model

progress + preservation → (initial) syntactic model

KEY property: universe of semantic relations

Short description

Our calculus:
System $F\omega$
- $+$ recursive types
- $+$ type equalities
- $+$ optional additional type constructors

Can be perceived as an IR used by a compiler.

Syntax

$$
\begin{array}{llll}
\text{kinds} & \kappa & ::= & \mathsf{T} \mid \kappa \Rightarrow \kappa \\
\text{constructors} & \mathsf{c} & ::= & \forall_\kappa \mid \exists_\kappa \mid \mu_\kappa \mid \to \mid \times \mid + \mid \mathsf{unit} \mid \mathsf{void} \\
\text{constraints} & \chi & ::= & \sigma \equiv_\kappa \tau \\
\text{types} & \tau, \sigma & ::= & \alpha \mid \lambda\alpha :: \kappa.\ \tau \mid \sigma\ \tau \mid \mathsf{c} \mid \chi \to \tau \mid \chi \times \tau
\end{array}
$$

Syntax

kinds $\kappa ::= \mathsf{T} \mid \kappa \Rightarrow \kappa$

constructors $c ::= \forall_\kappa \mid \exists_\kappa \mid \mu_\kappa \mid {\to} \mid \times \mid + \mid \mathsf{unit} \mid \mathsf{void}$

constraints $\chi ::= \sigma \equiv_\kappa \tau$

types $\tau, \sigma ::= \alpha \mid \lambda\alpha :: \kappa.\ \tau \mid \sigma\ \tau \mid c \mid \chi \to \tau \mid \chi \times \tau$

values $v ::= x \mid \langle\rangle \mid \lambda x.\ e \mid \langle v_1, v_2\rangle \mid \mathsf{inj}_1\ v \mid \mathsf{inj}_2\ v \mid \Lambda.\ e$
$\mid \mathsf{pack}\ v \mid \mathsf{roll}\ v \mid \lambda\bullet.\ e \mid \langle\bullet, v\rangle$

expressions $e ::= \mathsf{let}\ x = e_1\ \mathsf{in}\ e_2 \mid v_1\ v_2 \mid \mathsf{proj}_1\ v \mid \mathsf{proj}_2\ v$
$\mid \mathsf{case}\ v\ [x.\ e_1 \mid y.\ e_2] \mid v \mid \mathsf{abort}\ \bullet \mid v\ \bullet$
$\mid \mathsf{let}\ (\bullet, x) = v\ \mathsf{in}\ e$

eval. contexts $E ::= \square \mid \mathsf{let}\ x = E\ \mathsf{in}\ e$

- Type constructors are built-in functions on types.
- Constraint types are 'assert's and 'assume's for type equalities.

Dynamic semantics

Constraints don't have any interesting operational semantics, and their introduction and elimination forms just 'guide typechecking'.

$$(\lambda \bullet .\ e)\ \bullet \ \mapsto\ e$$

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$$\mathsf{let}\,(\bullet, x) = \langle \bullet, v\rangle\,\mathsf{in}\,e \mapsto\ e[v/x]$$

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$$\mathsf{let}\,(\bullet, x) = \langle \bullet, v\rangle\ \mathsf{in}\ e\ \mapsto\ e[v/x]$$

Informally, we can consider constraints as singleton types.

Discriminability

For impossible case elimination it is enough to look at the head symbols.

$$\frac{c_1 \neq c_2 \qquad (\Delta \vdash c_i\ \overline{\tau}_i :: \kappa)_{i \in \{1,2\}}}{\Delta \Vdash c_1\ \overline{\tau}_1 \#_{\kappa}\ c_2\ \overline{\tau}_2}$$

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$$\frac{\Delta \vdash \tau_1 :: \mathsf{T} \qquad \Delta \vdash \tau_2 :: \mathsf{T} \qquad \Delta \vdash \sigma_1 :: \mathsf{T} \qquad \Delta \vdash \sigma_2 :: \mathsf{T}}{\Delta \Vdash \tau_1 + \sigma_1 \#_{\mathsf{T}}\ \tau_2 \times \sigma_2}$$

Provability

The main crux of the system — injectivity. This rule makes it possible to use GADTs (e.g., in case of well-typed terms) to derive non-trivial equalities.

$$\frac{c :: (\kappa_i \Rightarrow)_i\, \kappa \qquad \Delta \mid \Phi \Vdash c\ (\sigma_i)_i \equiv_\kappa c\ (\tau_i)_i}{(\Delta \mid \Phi \Vdash \sigma_i \equiv_{\kappa_i} \tau_i)_i}$$

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$$\frac{\Delta \mid \Phi \Vdash \sigma_1 \times \tau_1 \equiv_{\mathsf{T}} \sigma_2 \times \tau_2}{\Delta \mid \Phi \Vdash \tau_1 \equiv_{\mathsf{T}} \tau_2}$$

Static semantics

We handle constraint passing manually to simplify semantic model.
This is handled by the typechecker in real calculi with GADTs.

$$\frac{\Delta \vdash \chi \text{ constr} \qquad \Delta \mid \Phi, \chi \mid \Gamma \vdash e : \tau}{\Delta \mid \Phi \mid \Gamma \vdash \lambda \bullet.\ e : \chi \to \tau} \qquad \frac{\Delta \mid \Phi \Vdash \chi \qquad \Delta \mid \Phi \mid \Gamma \vdash v : \tau}{\Delta \mid \Phi \mid \Gamma \vdash \langle \bullet, v \rangle : \chi \times \tau}$$

$$\frac{\Delta \mid \Phi \Vdash \sigma_1 \equiv_\kappa \sigma_2 \qquad \Delta \Vdash \sigma_1 \#_\kappa \sigma_2 \qquad \Delta \vdash \tau :: \mathsf{T}}{\Delta \mid \Phi \mid \Gamma \vdash \text{abort} \bullet : \tau}$$

$$\frac{\Delta \mid \Phi \Vdash \tau_1 \equiv_{\mathsf{T}} \tau_2 \qquad \Delta \mid \Phi \mid \Gamma \vdash e : \tau_1}{\Delta \mid \Phi \mid \Gamma \vdash e : \tau_2}$$

Type-indexed vectors

Some classical examples of dependent types can be expressed with just GADTs, if we use them as 'tags'.

$$
\begin{aligned}
&\mathsf{natvec} :: \mathsf{T} \Rightarrow \mathsf{T}\\
&\mathsf{natvec} \triangleq\\
&\quad \mu\varphi :: \mathsf{T} \Rightarrow \mathsf{T}.\ \lambda\alpha :: \mathsf{T}.\\
&\quad ((\alpha \equiv_{\mathsf{T}} \mathsf{void}) \times \mathsf{unit})\\
&\quad +(\mathbb{N} \times \exists\beta :: \mathsf{T}.\ (\alpha \equiv_{\mathsf{T}} (\beta + \mathsf{unit})) \times (\varphi\ \beta))
\end{aligned}
$$

natvec is either unit (and has void as its index)
or not unit (and the tail has a smaller index).

$$
\begin{aligned}
&\mathsf{nenatvec} :: \mathsf{T}\\
&\mathsf{nenatvec} \triangleq \exists\alpha :: \mathsf{T}.\ \mathsf{natvec}\ (\alpha + \mathsf{unit})
\end{aligned}
$$

Type-indexed vectors

The head function is now total! (We can eliminate the impossible case.)

$$
\begin{array}{l}
\mathsf{vhead} : \mathsf{nenatvec} \to \mathbb{N} \\
\mathsf{vhead}\, xs \triangleq \\
\quad \mathsf{let}\,(*, ys) = xs\,\mathsf{in} \\
\quad \mathsf{case}\ \mathsf{unroll}\, ys \\
\quad \mid \mathsf{inj}_1\,(\bullet, w).\ \mathsf{abort}\,\bullet \\
\quad \mid \mathsf{inj}_2\,\langle y, \_\rangle.\ y
\end{array}
$$

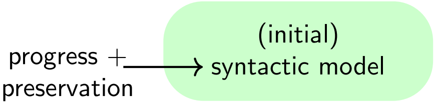
Enough about the syntax of our calculus! Can we actually prove something about it?

Naïve approach

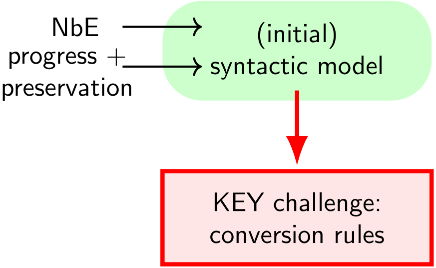
- Types are interpreted as sets of values. Constraints are interpreted as equalities of these sets.
- We can't validate injectivity rules, e.g., consider this instance:

$$\frac{\Delta \mid \Phi \Vdash \mathsf{void} \times \tau_1 \equiv_{\mathsf{T}} \mathsf{void} \times \tau_2}{\Delta \mid \Phi \Vdash \tau_1 \equiv_{\mathsf{T}} \tau_2}$$

- If $\emptyset \times A = \emptyset \times B$, then it isn't necessarily true that $A = B$.



progress + preservation $\longrightarrow$ (initial) syntactic model

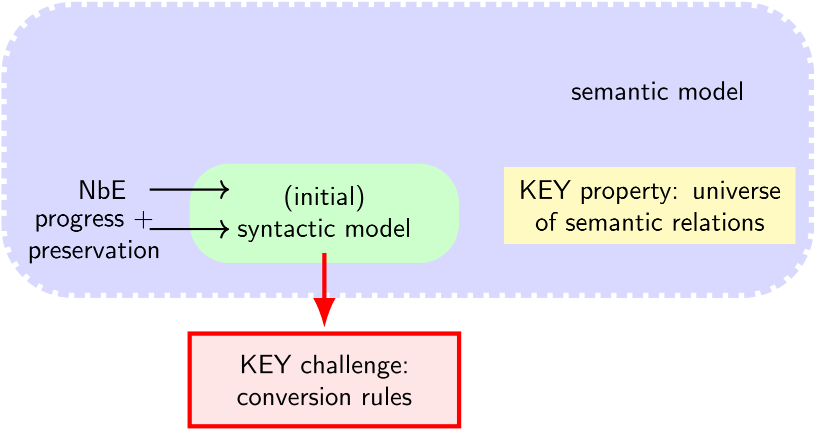


NbE

progress + preservation

(initial) syntactic model

KEY challenge: conversion rules



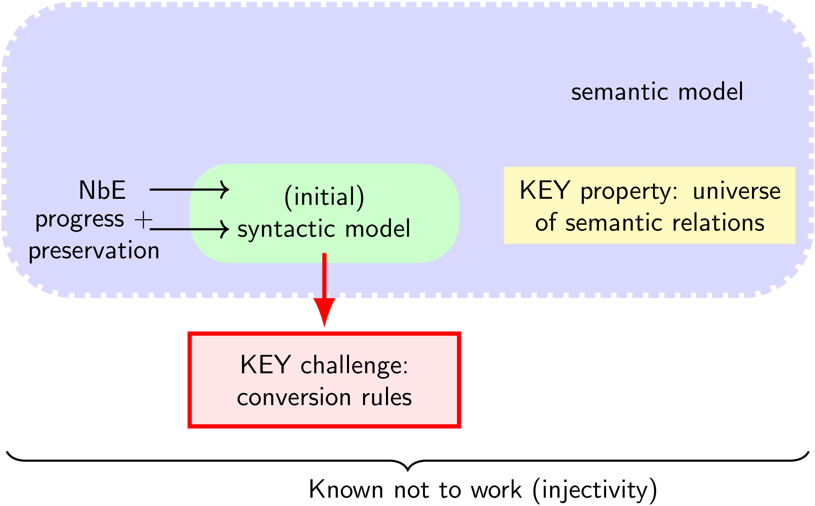
semantic model

NbE ⟶ (initial) syntactic model

progress + preservation ⟶ (initial) syntactic model

KEY property: universe of semantic relations

KEY challenge: conversion rules



semantic model

NbE

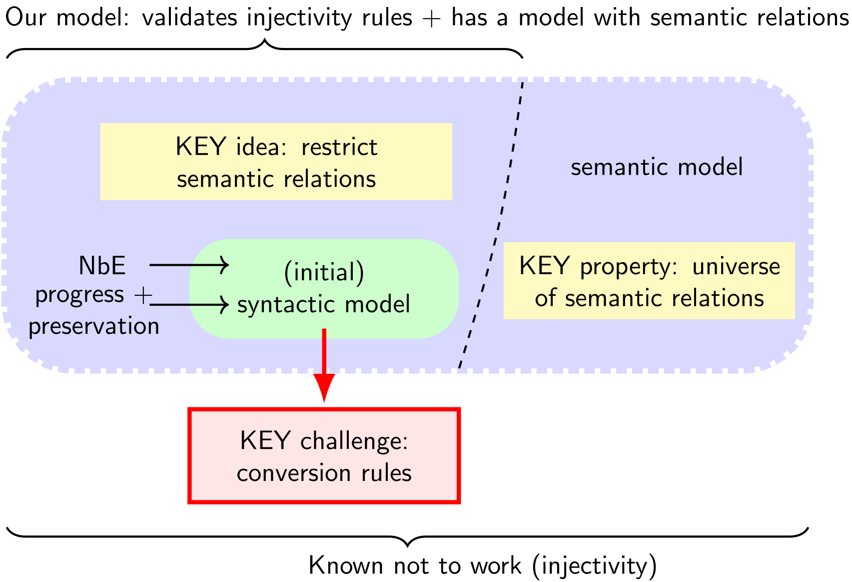
progress + preservation

(initial) syntactic model

KEY property: universe of semantic relations

KEY challenge: conversion rules

Known not to work (injectivity)



Our model: validates injectivity rules + has a model with semantic relations

KEY idea: restrict semantic relations

semantic model

NbE
progress + preservation

(initial) syntactic model

KEY property: universe of semantic relations

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Known not to work (injectivity)

High-level explanation

Idea: two stages.

- Before we compute the actual semantics, we need to do some equalities house-keeping.
- The first stage helps to reason about equalities.
- The second stage is for sets of values.

Normal forms

- We need an inductively defined universe of ‘codes’ for types.
- We can use NbE for types, and their normal forms as codes.

First stage

- *Normalization by evaluation* for types.
- Syntax of normal and neutral forms for types.
- Normalization is performed by composition of reify and eval.

$$\begin{aligned} [\![\mathsf{T}]\!] &\triangleq \mathrm{Neu}_{\mathsf{T}} \\ [\![\kappa_a \Rightarrow \kappa_r]\!] &\triangleq [\![\kappa_a]\!] \Rightarrow [\![\kappa_r]\!] \\ [\![\Delta]\!] &\triangleq \prod_{\alpha::\kappa \in \Delta} [\![\kappa]\!] \end{aligned}$$

$$\begin{aligned} \mathrm{reify} &: [\![\kappa]\!] \Rightarrow \mathrm{Nf}_{\kappa} \\ \mathrm{reflect} &: \mathrm{Neu}_{\kappa} \Rightarrow [\![\kappa]\!] \\ \mathrm{eval} &: \mathrm{Ty}^{\Delta}_{\kappa} \to ([\![\Delta]\!] \Rightarrow [\![\kappa]\!]) \end{aligned}$$

Setup for the second stage

We used step-indexed logic for this version of the calculus.
Language features might require additional gadgets.

$$\tau ::= T \mid \mathit{Val} \mid \mathit{Expr} \mid \mathrm{Prop} \mid 1 \mid \tau + \tau \mid \tau \times \tau \mid \tau \to \tau$$

$$\begin{aligned} t, P ::= {} & x \mid v \mid e \mid F(t_1, \ldots, t_n) \mid \\ & () \mid (t, t) \mid \pi_i\ t \mid \lambda x : \tau.\, t \mid t(t) \mid \\ & \mathrm{inl}\ t \mid \mathrm{inr}\ t \mid \mathrm{case}(t, x.t, y.t) \mid \\ & \mathrm{False} \mid \mathrm{True} \mid t =_\tau t \mid P \Rightarrow P \mid P \wedge P \mid P \vee P \mid \\ & \exists x : \tau.\, P \mid \forall x : \tau.\, P \mid \triangleright P \mid \mu\, x : \tau.\, t \end{aligned}$$

$$\frac{\Gamma, x : \tau \vdash t : \tau \qquad x \text{ is guarded in } t}{\Gamma \vdash \mu\, x : \tau.\, t : \tau}$$

Second stage

- We can interpret normal forms now, instead of arbitrary types.
- Syntactic equality of normal forms for constraints.

$$(\Delta \vdash \tau \equiv_\kappa \sigma \ \mathsf{constr})\,\mathrm{true} \triangleq$$
$$\forall(\Delta' : \mathrm{Ctx})(\eta : [\![\Delta]\!]^{\Delta'}).\mathrm{reify}(\mathrm{eval}(\tau)(\eta)) = \mathrm{reify}(\mathrm{eval}(\sigma)(\eta))$$

$$\mathcal{R}(\chi \times \nu)(\mathsf{v}) \triangleq \exists \mathsf{v}'.\mathsf{v} = \langle \bullet, \mathsf{v}' \rangle \wedge \chi\,\mathrm{true} \wedge \mathcal{R}(\nu)(\mathsf{v}')$$

But what happens when we want to go under binders in types
(at this point, it's either forall or recursive types)?

$$\mathcal{R}(\forall\alpha :: \kappa.\ \tau)(v) \triangleq \exists e.\ v = \Lambda.\ e \wedge \forall\mu \in ?.\mathrm{wp}(\mathcal{R}(\tau + \text{something about } \mu \text{ and } \alpha))(e)$$

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What if we want to validate an equality in τ that involves α?

If we try to use predicates, we end up with a function from an interpretation of $\mathbb{N}$ to some arbitrary interpretation of α and a syntactic constraint.

$$\mathcal{R}(\forall\alpha :: \mathsf{T}.\ \forall\beta :: \mathsf{T}.\ (\alpha \times \beta \equiv_{\mathsf{T}} \mathbb{N} \times \beta) \to \mathbb{N} \to \alpha)$$

Second stage*

- We cannot directly use predicates for ∀.
- Guarded recursion not only in the case of recursive types, but also in the case of ∀ (syntactic substitution strikes back).
- We can interpret normal forms now, instead of arbitrary types.
- Syntactic equality of normal forms for constraints.

$$\mathcal{R}(\forall\alpha :: \kappa.\ \tau)(v) \triangleq \exists e.\ v = \Lambda.\ e \wedge \forall\mu \in [\![\kappa]\!](\cdot).\triangleright\mathrm{wp}(\mathcal{R}(\mathrm{eval}(\tau)([\alpha \mapsto \mu])))(e)$$

$$\mathcal{R}(\chi \times \nu)(v) \triangleq \exists v'.v = \langle \bullet, v' \rangle \wedge \chi\,\mathrm{true} \wedge \mathcal{R}(\nu)(v')$$

Dealing with equalities

$$(\Delta \vdash \tau \equiv_\kappa \sigma \ \mathsf{constr}) \, \mathrm{true} \triangleq$$
$$\forall(\Delta' : \mathrm{Ctx})(\eta : [\![\Delta]\!]^{\Delta'}).\mathrm{reify}(\mathrm{eval}(\tau)(\eta)) = \mathrm{reify}(\mathrm{eval}(\sigma)(\eta))$$

- To validate the conversion rule we need reification to be injective.
- That's what *good* stands for.

Lemma (Injectivity of reify)

For any types τ_1, τ_2 *of kind* κ*, well-formed in* Δ
and any good environment $\eta : [\![\Delta]\!]^{\Delta'}$,
if $\mathrm{reify}(\mathrm{eval}(\tau_1)(\eta)) = \mathrm{reify}(\mathrm{eval}(\tau_2)(\eta))$*, then* $\mathrm{eval}(\tau_1)(\eta) = \mathrm{eval}(\tau_2)(\eta)$.

Second stage**

- We need to maintain a *good* environment.
- We cannot use purely semantic predicates in ∀.
- Guarded recursion not only in case of recursive types, but also in ∀ (syntactic substitution strikes back).
- We can interpret normal forms now, instead of arbitrary types.
- Syntactic equality of normal forms for constraints.

$$\mathcal{R}(\forall\alpha :: \kappa.\ \tau)(v) \triangleq \exists e.\ v = \Lambda.\ e \wedge \forall\mu \in [\![\kappa]\!](\cdot).\ \mathrm{good}(\mu) \rightarrow \triangleright\mathrm{wp}(\mathcal{R}(\mathrm{eval}(\tau)([\alpha \mapsto \mu])))(e)$$

$$\mathcal{R}(\chi \times \nu)(v) \triangleq \exists v'.v = \langle\bullet, v'\rangle \wedge \chi\,\mathrm{true} \wedge \mathcal{R}(\nu)(v')$$

Logical relation

We avoided a few problems so far:

- If the interpretation of equalities is too *semantical*, we cannot validate injectivity rules.
- If we use equalities of normal forms to interpret equalities, but do not use *good* environments, we cannot validate the conversion rules.

After that, we can actually validate all the rules, given a *good* environment for types, and valid contexts for constraints and term variables.

You might have a question: it's good, but is it usable for anything apart from proving safety in a semantical way?

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No!

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No!

But we can cook up something.

We can extend the syntax with arbitrary stuff at the base kind (remember, that reify and reflect are ‘inert’ for the base kind).

$$\frac{\varphi : X}{\varphi : \mathrm{Neu}^{\mathsf{\Delta}}_{\mathsf{T}}}$$

If X is instantiated with predicates (or relations), we can verify syntactically not well-typed programs (or prove interesting binary properties).

To evaluate the model we prove representation independence of type-indexed vectors and lists.

Moreover, we don't break anything in the process (e.g., we still can prove free theorems).

Limitations

- The model is not robust enough to ensure termination of the sub-calculus with restricted injectivity and no recursive types.
- Relational reasoning is limited to types at the base kind (well, in any case, programmers don't use types at higher kinds).

Contributions:

- Calculus for studies of GADTs.
- Novel approach to study semantics of feature-rich languages with syntactic constraints for types.
- Semantical models of a language that allows us to express GADTs:
  - Unary model that validates potential extensions for languages with GADTs.
  - Binary model that allows reasoning about representation independence (and (sic!) doesn't break anything from System F).

and future:

- Extensions (general effects).
- Relational interpretation of $\forall$ quantified at higher kinds. (Less restrictive interpretation of $\forall$?)
- The end goal is to provide a setup that can be used for designing a language that supports GADTs.

Placeholder before backup slides

[redacted]

Logical relation

$$\begin{aligned}
[\![\Phi]\!]_\eta\,\mathrm{true} &\triangleq \forall \varphi \in \Phi.\ [\![\varphi]\!]_\eta\,\mathrm{true} \\
[\![\Gamma]\!]_\eta &\triangleq \{\gamma \in \mathrm{dom}(\Gamma) \to \mathrm{Val} \mid \forall x \in \mathrm{dom}(\Gamma).\ \mathcal{R}(\mathrm{eval}(\Gamma(x))(\eta))(\gamma(x))\} \\
\Delta \mid \Phi \mid \Gamma \models e : \tau &\triangleq \forall \eta \in [\![\Delta]\!](\cdot).\ \mathrm{good}(\eta) \to [\![\Phi]\!]_\eta\,\mathrm{true} \to \forall \gamma \in [\![\Gamma]\!]_\eta \to \mathrm{wp}(\mathcal{R}(\mathrm{eval}(\tau)(\eta)))(e)
\end{aligned}$$

Injectivity and Cantor's paradox

Injectivity of some constructors implies false. It's a known fact, but can come up as a surprise.

For any injective constructor $c :: (\mathsf{T} \Rightarrow \mathsf{T}) \Rightarrow \mathsf{T}$ and type $\alpha :: \mathsf{T}$ it is possible to derive a value of type void in System $\mathsf{F}_{\omega}^{=i}$.

[redacted]

Non-termination

For any injective constructor $c :: (\mathsf{T} \Rightarrow \mathsf{T}) \Rightarrow \mathsf{T}$ and type $\alpha :: \mathsf{T}$ it is possible to derive a value of type void in System $\mathsf{F}_{\omega}^{=i}$.

- $\tau_c^{\mathrm{loop}} \triangleq \exists\beta :: \mathsf{T} \Rightarrow \mathsf{T}.\ (c\ \beta \equiv_{\mathsf{T}} \alpha) \times (\beta\ \alpha \rightarrow \mathsf{void})$
- $v^{\mathrm{loop}} \triangleq \lambda x.\ \mathsf{let}\,(*, (\bullet, y)) = x\,\mathsf{in}\,y\ (\mathsf{pack}\,\langle \bullet, y\rangle)$
- $\vdash v^{\mathrm{loop}} : \tau_c^{\mathrm{loop}}[(c\ (\lambda\alpha :: \mathsf{T}.\ \tau_c^{\mathrm{loop}}))/\alpha] \rightarrow \mathsf{void},$
- $\vdash v^{\mathrm{loop}}\ (\mathsf{pack}\,\langle \bullet, v^{\mathrm{loop}}\rangle) : \mathsf{void}$

[redacted]

Syntactic type-safety via NbE

Lemma (Consistency)

A discriminable constraint is not provable in an empty context: in other words, $\emptyset \mid \emptyset \Vdash \tau_1 \equiv_\kappa \tau_2$ *and* $\emptyset \Vdash \tau_1 \#_\kappa \tau_2$ *are contradictory.*

- Consequence of the injectivity of reify.
- Allows to discharge impossible cases.

Lemma (Canonical form for arrows)

If v *is a closed value of type* τ *and* τ *is provably equal to some arrow type in an empty context, then* v *is a lambda-abstraction with a well-typed body.*

$$(\emptyset \mid \emptyset \Vdash \tau \equiv_{\mathsf{T}} (\tau_1 \to \tau_2)) \wedge (\emptyset \mid \emptyset \mid \Gamma \vdash v : \tau)$$
$$\implies (\exists x e.\ v = \lambda x.\ e \wedge \emptyset \mid \emptyset \mid \Gamma, x : \tau_1 \vdash e : \tau_2)$$

Orthogonal extensions

References and concurrency.

$$\begin{array}{lcll}
\text{constructors} & c & ::= & \cdots \mid \mathsf{ref} \\
\text{references} & \ell & ::= & \mathbb{N} \\
\text{values} & v & ::= & \cdots \mid \ell \\
\text{expressions} & e & ::= & \cdots \mid \mathsf{fork}\ e \mid \mathsf{alloc}\ v \mid v := v \mid !\ v
\end{array}$$

The first stage stays the same, and the rest depends only on the logic used for defining $\mathcal{R}$.

The only requirements are that new effects should be expressed by type constructors, and that the ambient logic can express them.

[redacted]

Type-safe red-black trees

```
data Red
data Black
data Tree a where
  Tree :: Node Black n a -> Tree a

data Node t n a where
  Nil :: Node Black Zero a
  BlackNode :: NodeH t0 t1 n a -> Node Black (Succ n) a
  RedNode :: NodeH Black Black n a -> Node Red n a
data NodeH l r n a = NodeH (Node l n a) a (Node r n a)
```

Stronger type invariants.

Well-typed lambda terms

$$
\begin{aligned}
&\mathsf{Tm} :: \mathsf{T} \Rightarrow \mathsf{T}\\
&\mathsf{Tm} \triangleq\\
&\quad \mu\varphi :: \mathsf{T} \Rightarrow \mathsf{T}.\ \lambda\alpha :: \mathsf{T}.\\
&\quad \alpha + (\exists\beta,\gamma :: \mathsf{T}.\ (\alpha \equiv_{\mathsf{T}} (\beta \to \gamma)) \times (\beta \to \varphi\ \gamma))\\
&\qquad + (\exists\beta :: \mathsf{T}.\ \varphi\ (\beta \to \alpha) \times \varphi\ \beta)
\end{aligned}
$$

[redacted]

Well-typed lambda terms

$$
\begin{array}{l}
\mathsf{eval} : \forall\alpha :: \mathsf{T}.\ \mathsf{Tm}\ \alpha \to \alpha \\
\mathsf{eval} \triangleq \\
\quad \mathsf{fix}\, \lambda f.\ \Lambda.\ \lambda x. \\
\quad \mathsf{case}\ \mathsf{unroll}\, x \\
\quad \mid \mathsf{inj}_1\, y.\ y \\
\quad \mid \mathsf{inj}_2\, y.\ \mathsf{case}\ y \\
\qquad \mid \mathsf{inj}_1\, (*, (*, (\bullet, g))).\ \lambda z.\ f * (g\ z) \\
\qquad \mid \mathsf{inj}_2\, (*, \langle g, x \rangle).\ (f * g)\ (f * x)
\end{array}
$$

[redacted]

Logical relation for denotations

For any two bigger related contexts and arguments in this extended contexts, results are related after extension.

$$\begin{aligned}\eta \mid \nu_1 \approx_{\mathsf{T}} \nu_2 &\triangleq [\![\nu_1]\!]_\eta = \nu_2\\ \eta \mid \varphi_1 \approx_{\kappa_a \Rightarrow \kappa_r} \varphi_2 &\triangleq \forall \Delta_1', \Delta_2', (\delta_1 : \mathsf{hom}_{\mathcal{K}}(\Delta_1', \Delta_1), \delta_2 : \mathsf{hom}_{\mathcal{K}}(\Delta_2', \Delta_2)), (\eta' : [\![\Delta_1']\!]^{\Delta_2'}), \mu_1, \mu_2.\\ &\qquad (\delta_2^* \eta = \lambda \mathsf{x}.\ \eta'(\delta_1(\mathsf{x}))) \to (\eta' \mid \mu_1 \approx_{\kappa_a} \mu_2) \to (\eta' \mid \varphi_1(\delta_1, \mu_1) \approx_{\kappa_r} \varphi_2(\delta_2, \mu_2))\end{aligned}$$

Lemma

If $\eta \mid \mu_1 \approx \mu_2$, *then* $[\![\mathrm{reify}(\mu_1)]\!]_\eta = \mu_2$.
If $\eta \mid \eta_1 \approx \eta_2$, *then* $\eta \mid [\![\tau]\!]_{\eta_1} \approx [\![\tau]\!]_{\eta_2}$.

$$\imath \mid \nu \approx_\kappa \nu$$