Verification of Chase-Lev work-stealing deque

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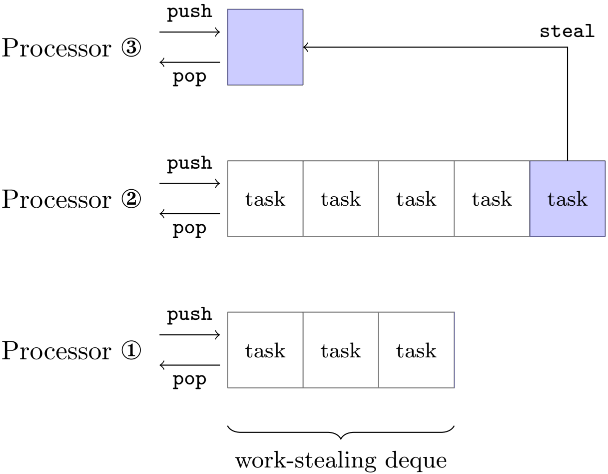
Context: scheduler for task-based parallelism

- Cilk (C, C++)
- Threading Building Blocks (C++)
- Taskflow (C++)
- Tokio (Rust)
- Goroutines (Go)
- Domainslib (OCaml 5)

Work-stealing



Chase-Lev work-stealing deque

1. *The Implementation of the Cilk-5 Multithreaded Language.*
   Frigo, Leiserson & Randall (1998).
   - lock
2. *Thread Scheduling for Multiprogrammed Multiprocessors.*
   Arora, Blumofe & Plaxton (1998).
   - non-blocking
   - one fixed size array, potential overflow
3. *A dynamic-sized nonblocking work stealing deque.*
   Hendler, Lev, Moir, & Shavit (2004).
   - non-blocking
   - list of small arrays, no overflow
4. *Dynamic circular work-stealing deque.*
   Chase & Lev (2005).
   - non-blocking
   - circular arrays, no overflow

Why is it interesting?

- Demonstration of Iris on a (simplified) real-life concurrent data structure.
- Rich ghost state to enforce a subtle protocol.
  - logical state $\neq$ physical state
  - external future-dependent linearization point
- Nontrivial use of prophecy variables.

The rest of this talk

- Specification using logically atomic triples.
- Rough idea of how the data structure works.
- Why we need prophecy variables.

Specification

Physical state

Logical state

Prophecy variables

Specification — `chaselev_make`

$$\frac{\{\,\text{True}\,\}}{\dfrac{\texttt{chaselev\_make ()}}{\left\{\lambda\, t.\ \ \text{chaselev-inv}\ t\ \iota\ \ast\ \text{chaselev-model}\ t\ []\ \ast\ \text{chaselev-owner}\ t\right\}}}$$

Specification — `chaselev_make`

$$\frac{\{\,\text{True}\,\}}{\texttt{chaselev\_make ()}}$$

$$\left\{ \lambda\, t.\ \ \text{chaselev-inv}\ t\ \iota\ \ast\ \text{chaselev-model}\ t\ []\ \ast\ \text{chaselev-owner}\ t \right\}$$

t is an instance of Chase-Lev deque.
Enforces a protocol (using an Iris invariant).

Specification — `chaselev_make`

$$\frac{\{\,\text{True}\,\}}{\frac{\texttt{chaselev\_make ()}}{\left\{\lambda\, t.\ \ \text{chaselev-inv}\ t\ \iota\ \ *\ \ \text{chaselev-model}\ t\ []\ \ *\ \ \text{chaselev-owner}\ t\right\}}}$$

Asserts the list of values that t logically contains.

Specification — `chaselev_make`

$$\frac{\{\,\text{True}\,\}}{\frac{\texttt{chaselev\_make ()}}{\left\{\lambda\, t.\;\; \text{chaselev-inv}\; t\; \iota \;*\; \text{chaselev-model}\; t\; [] \;*\; \text{chaselev-owner}\; t\right\}}}$$

Gives the owner exclusive access to his end of t.

Specification — `chaselev_push`

$$\left\{\ \text{chaselev-inv}\ t\ \iota\ \ast\ \text{chaselev-owner}\ t\ \right\}$$
$$\left\langle\ \forall\, vs.\ \ \text{chaselev-model}\ t\ vs\ \right\rangle$$
$$\texttt{chaselev\_push}\ t\ v,\ \uparrow\iota$$
$$\left\langle\ \exists.\ \ \text{chaselev-model}\ t\ (vs +\!\!+ [v])\ \right\rangle$$
$$\left\{\ \texttt{()}.\ \ \text{chaselev-owner}\ t\ \right\}$$

Specification — `chaselev_push`

Specification of a concurrent operation ($\simeq$ transaction):
standard triple + logically atomic triple

$$\frac{\frac{\frac{\frac{\{\, P \,\}}{\langle \forall\, \overline{x}.\ P_{\text{lin}} \rangle}}{e,\ \mathcal{E}}}{\langle \exists\, \overline{y}.\ Q_{\text{lin}} \rangle}}{\{\, res.\ Q \,\}}$$

P : private precondition

Q : private postcondition

P_{lin} : public precondition

Q_{lin} : public postcondition

Specification — chaselev_push

For a concurrent data structure:

$$\frac{\frac{\frac{\frac{\{\,\text{???-inv}\cdots * P\,\}}{\langle\,\forall\,\overline{x}.\ \text{???-model}\cdots\rangle}}{e,\ \mathcal{E}}}{\langle\,\exists\,\overline{y}.\ \text{???-model}\cdots\rangle}}{\{\,res.\ Q\,\}}$$

Specification — `chaselev_push`

$$\frac{\frac{\frac{\frac{\left\{\ \text{chaselev-inv}\ t\ \iota\ \ast\ \text{chaselev-owner}\ t\ \right\}}{\left\langle\ \forall\, vs.\ \ \text{chaselev-model}\ t\ vs\ \right\rangle}}{\texttt{chaselev\_push}\ t\ v,\ \uparrow\iota}}{\left\langle\ \exists\texttt{.}\ \ \text{chaselev-model}\ t\ (vs +\!\!+ [v])\ \right\rangle}}{\left\{\ \texttt{().}\ \ \text{chaselev-owner}\ t\ \right\}}$$

t is an instance of Chase-Lev deque.

Specification — `chaselev_push`

$$\left\{\ \text{chaselev-inv}\ t\ \iota\ \ast\ \text{chaselev-owner}\ t\ \right\}$$

$$\left\langle\ \forall\, vs.\ \ \text{chaselev-model}\ t\ vs\ \right\rangle$$

$$\texttt{chaselev\_push}\ t\ v,\ \uparrow\iota$$

$$\left\langle\ \exists\,.\ \ \text{chaselev-model}\ t\ (vs +\!\!+ [v])\ \right\rangle$$

$$\left\{\ ().\ \ \text{chaselev-owner}\ t\ \right\}$$

This operation is reserved to the owner of t.

Specification — `chaselev_push`

$$\left\{\ \text{chaselev-inv}\ t\ \iota\ *\ \text{chaselev-owner}\ t\ \right\}$$

$$\left\langle\ \forall\, vs.\ \ \text{chaselev-model}\ t\ vs\ \right\rangle$$

$$\texttt{chaselev\_push}\ t\ v,\ \uparrow\iota$$

$$\left\langle\ \exists.\ \ \text{chaselev-model}\ t\ (vs +\!\!+ [v])\ \right\rangle$$

$$\left\{\ ().\ \ \text{chaselev-owner}\ t\ \right\}$$

v is atomically pushed at the owner's end of t.

Specification — `chaselev_pop`

$$\left\{ \text{chaselev-inv } t\ \iota \ * \ \text{chaselev-owner } t \right\}$$

$$\left\langle \forall\, vs.\ \text{chaselev-model } t\ vs \right\rangle$$

$$\texttt{chaselev\_pop}\ t,\ \uparrow \iota$$

$$\left\langle \exists\, o.\ \bigvee \left[\begin{array}{l} vs = [\,] * o = \texttt{NONE} * \text{chaselev-model } t\ [\,] \\ \exists\, v, vs'.\ vs = vs' +\!\!+ [v] * o = \texttt{SOME}\ v * \text{chaselev-model } t\ vs' \end{array} \right] \right\rangle$$

$$\left\{ o.\ \text{chaselev-owner } t \right\}$$

Specification — `chaselev_pop`

$$\left\{\ \text{chaselev-inv}\ t\ \iota\ \ast\ \text{chaselev-owner}\ t\ \right\}$$

$$\left\langle\ \forall\, vs.\ \ \text{chaselev-model}\ t\ vs\ \right\rangle$$

$$\texttt{chaselev\_pop}\ t,\ \uparrow\iota$$

$$\left\langle\ \exists\, o.\ \bigvee\left[\begin{array}{l} vs = [\,] \ast o = \texttt{NONE} \ast \text{chaselev-model}\ t\ [\,] \\ \exists\, v, vs'.\ vs = vs' +\!\!+ [v] \ast o = \texttt{SOME}\ v \ast \text{chaselev-model}\ t\ vs' \end{array}\right]\ \right\rangle$$

$$\left\{\ o.\ \ \text{chaselev-owner}\ t\ \right\}$$

t is an instance of Chase-Lev deque.

Specification — `chaselev_pop`

$$\left\{ \text{chaselev-inv}\ t\ \iota\ \ast\ \text{chaselev-owner}\ t \right\}$$

$$\left\langle \forall\, vs.\ \ \text{chaselev-model}\ t\ vs \right\rangle$$

$$\texttt{chaselev\_pop}\ t,\ \uparrow \iota$$

$$\left\langle \exists\, o.\ \ \bigvee \left[\begin{array}{l} vs = [\,] \ast o = \texttt{NONE} \ast \text{chaselev-model}\ t\ [\,] \\ \exists\, v, vs'.\ vs = vs' +\!\!+ [v] \ast o = \texttt{SOME}\ v \ast \text{chaselev-model}\ t\ vs' \end{array} \right] \right\rangle$$

$$\left\{ o.\ \ \text{chaselev-owner}\ t \right\}$$

This operation is reserved to the owner of t.

Specification — `chaselev_pop`

$$\left\{\ \text{chaselev-inv}\ t\ \iota\ \ast\ \text{chaselev-owner}\ t\ \right\}$$

$$\left\langle\ \forall\, vs.\ \ \text{chaselev-model}\ t\ vs\ \right\rangle$$

$$\texttt{chaselev\_pop}\ t,\ \uparrow\iota$$

$$\left\langle\ \exists\, o.\ \ \bigvee\begin{bmatrix} vs = [\,] \ast o = \texttt{NONE} \ast \text{chaselev-model}\ t\ [\,] \\ \exists\, v, vs'.\ vs = vs' +\!\!+ [v] \ast o = \texttt{SOME}\ v \ast \text{chaselev-model}\ t\ vs' \end{bmatrix}\ \right\rangle$$

$$\left\{\ o.\ \ \text{chaselev-owner}\ t\ \right\}$$

Either 1) t is seen empty
or 2) some value v is atomically popped at the owner's end of t.

Specification — `chaselev_steal`

$$\cfrac{\left\{\ \text{chaselev-inv}\ t\ \iota\ \right\}}{\cfrac{\left\langle\ \forall\, vs.\ \ \text{chaselev-model}\ t\ vs\ \right\rangle}{\cfrac{\texttt{chaselev\_steal}\ t,\ \uparrow\iota}{\cfrac{\left\langle\ \exists\, o.\ \ \bigvee\left[\begin{array}{l} vs = [\,] * o = \texttt{NONE} * \text{chaselev-model}\ t\ [\,] \\ \exists\, v, vs'.\ vs = v :: vs' * o = \texttt{SOME}\ v * \text{chaselev-model}\ t\ vs' \end{array}\right]\ \right\rangle}{\left\{\, o.\ \text{True}\,\right\}}}}}$$

Specification — `chaselev_steal`

$$\left\{\ \text{chaselev-inv}\ t\ \iota\ \right\}$$

$$\left\langle\ \forall\, vs.\ \ \text{chaselev-model}\ t\ vs\ \right\rangle$$

$$\texttt{chaselev\_steal}\ t,\ \uparrow \iota$$

$$\left\langle\ \exists\, o.\ \ \bigvee \begin{bmatrix} vs = [\,] * o = \texttt{NONE} * \text{chaselev-model}\ t\ [\,] \\ \exists\, v, vs'.\ vs = v :: vs' * o = \texttt{SOME}\ v * \text{chaselev-model}\ t\ vs' \end{bmatrix}\ \right\rangle$$

$$\{\, o.\ \text{True}\,\}$$

t is an instance of Chase-Lev deque.

Specification — `chaselev_steal`

$$\left\{\ \text{chaselev-inv}\ t\ \iota\ \right\}$$

$$\left\langle\ \forall\, vs.\ \ \text{chaselev-model}\ t\ vs\ \right\rangle$$

$$\texttt{chaselev\_steal}\ t,\ \uparrow \iota$$

$$\left\langle\ \exists\, o.\ \ \bigvee \left[\begin{array}{l} vs = [] * o = \texttt{NONE} * \text{chaselev-model}\ t\ [] \\ \exists\, v, vs'.\ vs = v :: vs' * o = \texttt{SOME}\ v * \text{chaselev-model}\ t\ vs' \end{array}\right]\ \right\rangle$$

$$\left\{\ o.\ \text{True}\ \right\}$$

Either 1) t is seen empty
or 2) some value v is atomically popped at the thieves' end of t.

Specification

Physical state

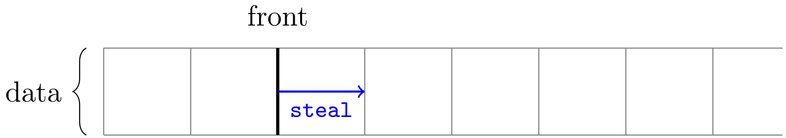
Logical state

Prophecy variables

Physical state

data: infinite array storing all values

Physical state



data: infinite array storing all values

front: *monotone* index for thieves' end

Physical state

data

data: in

front: m

ChaselevFrontLbGet

$$\frac{\boxed{\bullet\ \mathit{front}}^{\gamma.\text{front}}}{\boxed{\circ\ \mathit{front}}^{\gamma.\text{front}}}$$

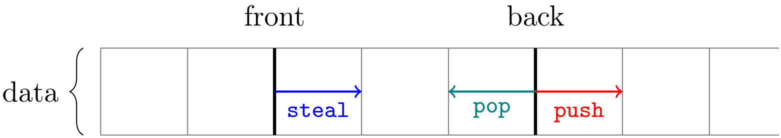
ChaselevFrontValid

$$\frac{\boxed{\bullet\ \mathit{front}_1}^{\gamma.\text{front}} \qquad \boxed{\circ\ \mathit{front}_2}^{\gamma.\text{front}}}{\mathit{front}_2 \leqslant \mathit{front}_1}$$

ChaselevFrontUpdate

$$\frac{\mathit{front} \leqslant \mathit{front}' \qquad \boxed{\bullet\ \mathit{front}}^{\gamma.\text{front}}}{\boxed{\bullet\ \mathit{front}'}^{\gamma.\text{front}}}$$

Physical state

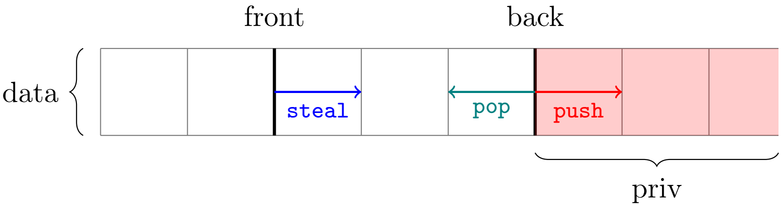


data: infinite array storing all values

front: *monotone* index for thieves' end

back: index for owner's end

Physical state



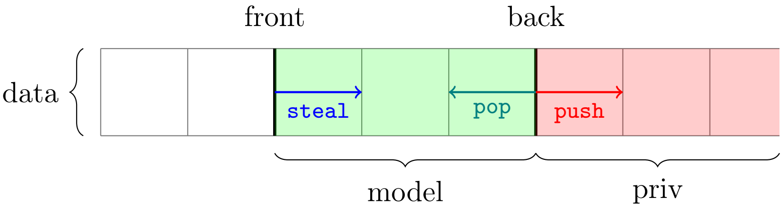
data: infinite array storing all values

front: *monotone* index for thieves' end

back: index for owner's end

priv: list of private values (controlled by owner)

Physical state



data: infinite array storing all values

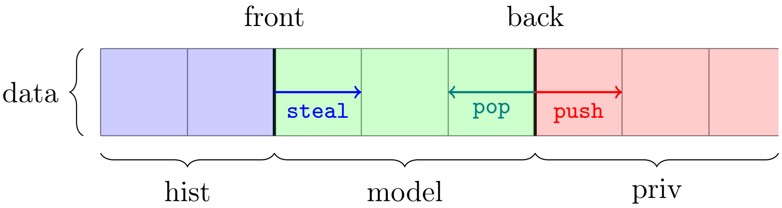
front: *monotone* index for thieves' end

back: index for owner's end

priv: list of private values (controlled by owner)

model: list of contained values

Physical state



data: infinite array storing all values

front: *monotone* index for thieves' end

back: index for owner's end

priv: list of private values (controlled by owner)

model: list of contained values

hist: *monotone* list of history values

Physical state

data

data: in

front: m

back: in

priv: lis

model: lis

hist: m

ChaselevHistLbGet

$$\frac{\boxed{\bullet\ \mathit{hist}}^{\gamma.\text{hist}}}{\boxed{\circ\ \mathit{hist}}^{\gamma.\text{hist}}}$$

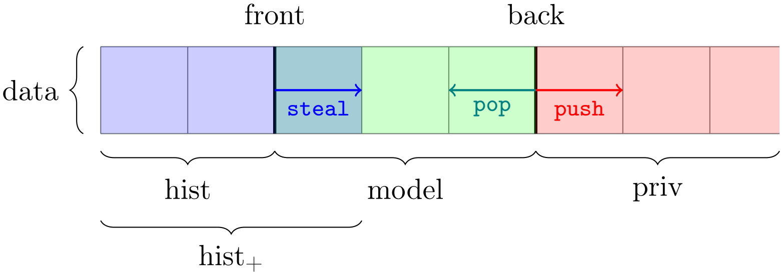
ChaselevHistValid

$$\frac{\boxed{\bullet\ \mathit{hist}_1}^{\gamma.\text{hist}} \qquad \boxed{\circ\ \mathit{hist}_2}^{\gamma.\text{hist}}}{\mathit{hist}_2 \sqsubseteq_{\text{prefix}} \mathit{hist}_1}$$

ChaselevHistUpdate

$$\frac{\boxed{\bullet\ \mathit{hist}}^{\gamma.\text{hist}}}{\boxed{\bullet\ (\mathit{hist} \mathbin{+\!\!+} [v])}^{\gamma.\text{hist}}}$$

Physical state



data: infinite array storing all values
front: *monotone* index for thieves' end
back: index for owner's end

priv: list of private values (controlled by owner)
model: list of contained values
hist: *monotone* list of history values
hist_+: *monotone* list of extended history values

Specification

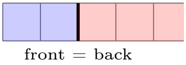
Physical state

Logical state

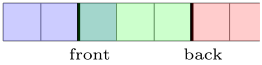
Prophecy variables

Logical state

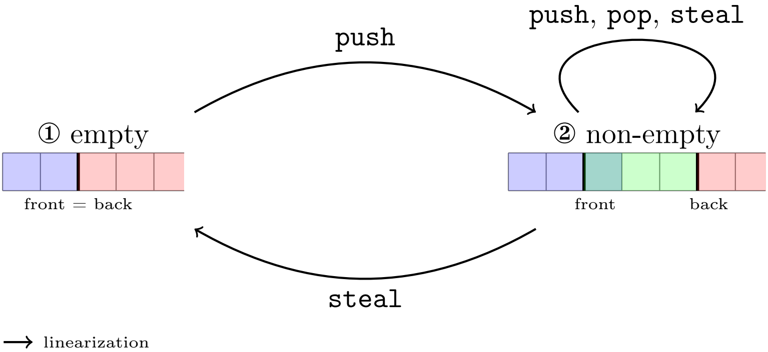
① empty



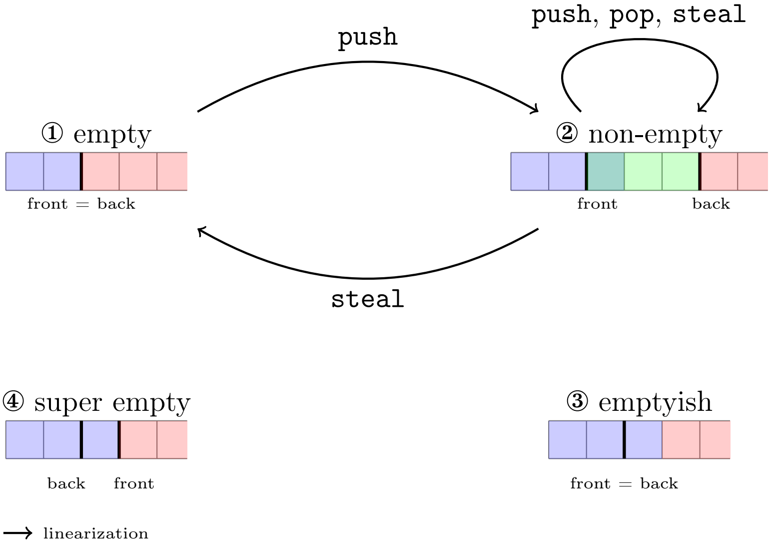
② non-empty



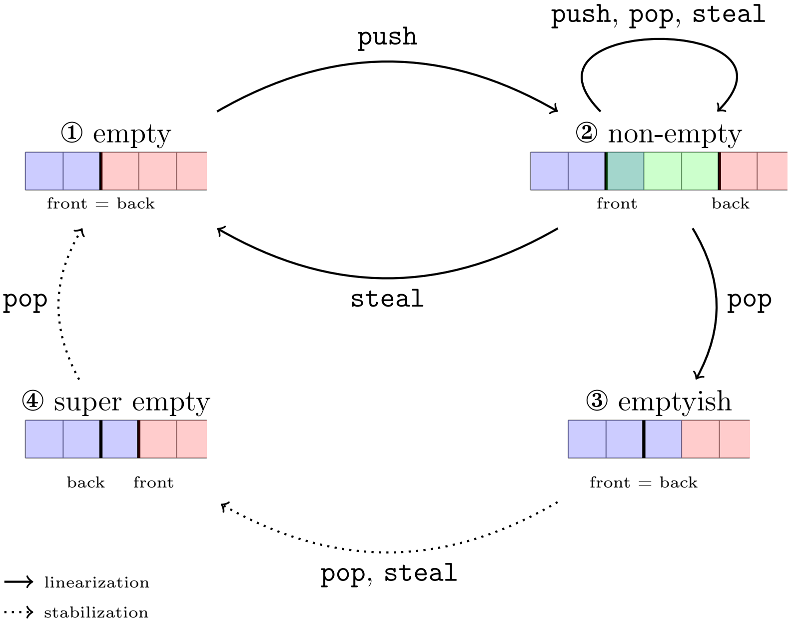
Logical state



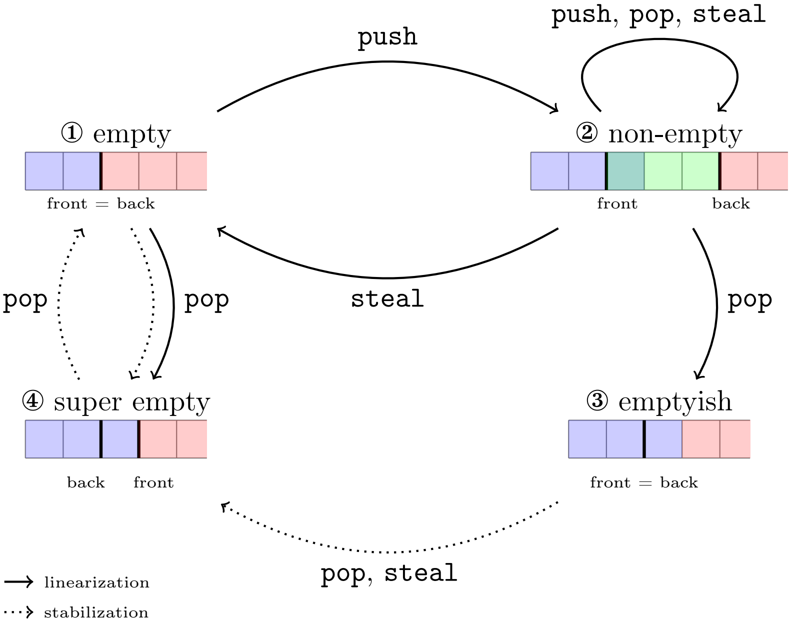
Logical state



Logical state



Logical state



Specification

Physical state

Logical state

Prophecy variables

Prophecy variables

The future is ours: prophecy variables in separation logic.
Jung, Lepigre, Parthasarathy, Rapoport, Timany, Dreyer & Jacobs (2020).

$$\{\,\mathrm{True}\,\}\ \texttt{NewProph}\ \{\,\lambda\, p.\ \exists\, \mathit{prophs}.\, \mathrm{proph}\ p\ \mathit{prophs}\,\}$$

$$\frac{\begin{array}{c}\text{atomic } e \\ \mathrm{proph}\ p\ \mathit{prophs} \\ \mathrm{WP}\ e\ \left\{\begin{array}{l}\lambda w.\ \forall\, \mathit{prophs}'. \\ \mathit{prophs} = (w, v) :: \mathit{prophs}' \twoheadrightarrow \\ \mathrm{proph}\ p\ \mathit{prophs}' \twoheadrightarrow \\ \Phi\ w\end{array}\right\}\end{array}}{\mathrm{WP}\ \texttt{Resolve}\ e\ p\ v\ \{\,\Phi\,\}}$$

Back to *The future is ours* (Jung *et al.*)

```
let rdcss rm rn m1 n1 n2 =
  let p = NewProph in
  let descr = ref (rm, m1, n1, n2, p) in
  ...
```

```
let complete descr rn =
  let (rm, m1, n1, n2, p) = !descr in
  let id = NewId in
  let m = !rm in
  let n_new = if m = m1 then n2 else n1 in
  Resolve (CmpXchg rn (inr descr) (inl n_new)) p id ;
  ()
```

Prophecy variables with memory

$$\{\,\text{True}\,\}\ \texttt{NewProph}\ \{\,\lambda\, p.\ \exists\,\gamma, \mathit{prophs}.\ \text{proph}\ p\ \gamma\ []\ \mathit{prophs}\,\}$$

$$\frac{\begin{array}{c}\text{atomic } e \\ \text{proph } p\ \gamma\ \mathit{past}\ \mathit{prophs} \\ \text{WP } e\ \left\{\begin{array}{l}\lambda w.\ \forall\,\mathit{prophs}'.\\ \mathit{prophs} = (w, v) :: \mathit{prophs}' \twoheadrightarrow\!\!* \\ \text{proph } p\ \gamma\ (\mathit{past} +\!\!+ [(w, v)])\ \mathit{prophs}' \twoheadrightarrow\!\!* \\ \Phi\ w \end{array}\right\}\end{array}}{\text{WP } \texttt{Resolve}\ e\ p\ v\ \{\,\Phi\,\}}$$

Prophecy variables with memory

$$\frac{\text{proph } p\ \gamma\ \mathit{past}\ \mathit{prophs}}{\text{proph-lb } \gamma\ \mathit{prophs}}\ \textsc{ProphecyLbGet}$$

$$\frac{\text{proph } p\ \gamma\ \mathit{past}\ \mathit{prophs}_1 \qquad \text{proph-lb } \gamma\ \mathit{prophs}_2}{\exists\, \mathit{past}_1, \mathit{past}_2.\ \bigwedge \left[\begin{array}{l} \mathit{past} = \mathit{past}_1 +\!\!+\ \mathit{past}_2 \\ \mathit{past}_2 +\!\!+\ \mathit{prophs}_1 = \mathit{prophs}_2 \end{array} \right.}\ \textsc{ProphecyValid}$$

Conclusion

- Coq mechanization is available on `github` :
  `https://github.com/clef-men/caml5`

- Simplified Chase-Lev deque (one infinite array) ✓
  Real-life Chase-Lev deque (multiple circular arrays) 

- Proof looks more complex than the sketch. In particular, transitions between logical states are not really formalized.

- We plan to verify more primitives (Domainslib, Taskflow) based on Chase-Lev deque. This is thanks to modularity of Iris specifications.

Thank you for your attention!

Implementation — chaselev_make

```
let chaselev_make _ =
  let t = AllocN 4 () in
  t.front <- 0 ;
  t.back <- 0 ;
  t.data <- inf_array_make () ;
  t.prophecy <- NewProph ;
  t
```

Implementation — chaselev_push

```
let chaselev_push t v =
  let back = !t.back in
  inf_array_set !t.data back v ;
  t.back <- back + 1
```

Implementation — chaselev_steal

```
let rec chaselev_steal t =
  let id = NewId in
  let front = !t.front in
  let back = !t.back in
  if front < back then (
    if Snd (
      Resolve (
        CmpXchg t.front front (front + 1)
       ) !t.prophecy (front, id)
    ) then (
      SOME (inf_array_get !t.data front)
    ) else (
      chaselev_steal t
    )
  ) else (
    NONE
  )
```

Implementation — chaselev_pop

```
let chaselev_pop t =
  let id = NewId in
  let back = !t.back - 1 in
  t.back <- back ;
  let front = !t.front in
  if back < front then (
    t.back <- front
  ) else (
    if front < back then (
      SOME (inf_array_get !t.data back)
    ) else (
      if Snd (
        Resolve (
          CmpXchg t.front front (front + 1)
        ) !t.prophecy (front, id)
      ) then (
        t.back <- front + 1 ;
        SOME (inf_array_get !t.data back)
      ) else (
        t.back <- front + 1 ;
        NONE
      )
```

Infinite array

$$\frac{\{\,\text{True}\,\}}{\texttt{inf\_array\_make}\ v} \atop \{\,\lambda\ arr.\ \text{inf-array-model}\ arr\ (\lambda\_.\ v)\,\}$$

$$\frac{\langle\,\forall\, vs.\ \text{inf-array-model}\ arr\ \ vs * 0 \leqslant i\,\rangle}{\texttt{inf\_array\_get}\ arr\ i} \atop \langle\,\exists\,.\ vs\ i.\ \text{inf-array-model}\ arr\ \ vs\,\rangle$$

$$\frac{\langle\,\forall\, vs.\ \text{inf-array-model}\ arr\ \ vs * 0 \leqslant i\,\rangle}{\texttt{inf\_array\_set}\ arr\ i\ v} \atop \langle\,\exists\,.\ \_.\ \text{inf-array-model}\ arr\ \ vs[i \mapsto v]\,\rangle$$

Invariant

$$\text{chaselev-inv}\ t\ \iota \triangleq$$
$$\exists\, \ell, \gamma, \mathit{data}, p.$$
$$\ast \left[\begin{array}{l} t = \ell * \text{meta}\ \ell\ \gamma \\ \ell.\text{data} \mapsto_{\square} \mathit{data} * \ell.\text{prophecy} \mapsto_{\square} p \\ \boxed{\text{chaselev-inv-inner}\ \ell\ \gamma\ \iota\ \mathit{data}\ p}^{\iota} \end{array}\right.$$

Invariant

$$
\begin{aligned}
&\text{chaselev-inv-inner } \ell\ \gamma\ \iota\ \mathit{data}\ p \triangleq \\
&\ \exists\, \mathit{front}, \mathit{back}, \mathit{hist}, \mathit{model}, \mathit{priv}, \mathit{past}, \mathit{prophs}. \\
&\ast \left[\begin{array}{l}
\ell.\text{front} \mapsto \mathit{front} \ast \ell.\text{back} \mapsto \mathit{back} \\
\boxed{\bullet\,(\mathit{back}, \mathit{priv})}^{\gamma.\text{ctl}} \\
\boxed{\bullet\,\mathit{front}}^{\gamma.\text{front}} \\
\text{inf-array-model } \mathit{data}\ (\mathit{hist} +\!\!+\ \mathit{model})\ \mathit{priv} \\
\boxed{\bullet\,\mathit{model}}^{\gamma.\text{model}} \quad \ast\ |\mathit{model}| = (\mathit{back} - \mathit{front})_+ \\
\text{wise-prophet-model } p\ \gamma.\text{prophet}\ \mathit{past}\ \mathit{prophs} \\
\forall (\mathit{front}', \_) \in \mathit{past}.\ \mathit{front}' < \mathit{front} \\
\text{chaselev-state } \gamma\ \iota\ \mathit{front}\ \mathit{back}\ \mathit{hist}\ \mathit{model}\ \mathit{prophs}
\end{array}\right.
\end{aligned}
$$

State

$$\text{chaselev-state}\ \gamma\ \iota\ \mathit{front}\ \mathit{back}\ \mathit{hist}\ \mathit{model}\ \mathit{prophs} \triangleq$$

$$\bigvee \left[\begin{array}{l} \text{chaselev-state}_1\ \gamma\ \mathit{front}\ \mathit{back}\ \mathit{hist} \\ \text{chaselev-state}_2\ \gamma\ \iota\ \mathit{front}\ \mathit{back}\ \mathit{hist}\ \mathit{model}\ \mathit{prophs} \\ \text{chaselev-lock}\ \gamma * \bigvee \left[\begin{array}{l} \text{chaselev-state}_3\ \gamma\ \mathit{front}\ \mathit{back}\ \mathit{hist}\ \mathit{prophs} \\ \text{chaselev-state}_4\ \gamma\ \mathit{front}\ \mathit{back}\ \mathit{hist} \end{array} \right. \end{array} \right.$$

State 1 (empty)

$$\text{chaselev-state}_1 \; \gamma \; \mathit{front} \; \mathit{back} \; \mathit{hist} \triangleq$$

$$\ast \left[\begin{array}{l} \mathit{front} = \mathit{back} \\ \boxed{\bullet\, \mathit{hist}}^{\gamma.\text{hist}} \ast |\mathit{hist}| = \mathit{front} \\ \boxed{\bullet\, - \,.\; \circ\, -}^{\gamma.\text{winner}} \end{array} \right.$$

State 2 (non-empty)

$$
\text{chaselev-state}_2\ \gamma\ \iota\ \mathit{front}\ \mathit{back}\ \mathit{hist}\ \mathit{model}\ \mathit{prophs} \triangleq
\mathop{\ast}\left[\begin{array}{l}
\mathit{front} < \mathit{back} \\
\boxed{\bullet\,(\mathit{hist} +\!\!+ [\mathit{model}[0]])}^{\gamma.\mathrm{hist}} \ast |\mathit{hist}| = \mathit{front} \\
\quad \textbf{match}\ \text{filter}\ (\lambda(\mathit{front}', \_).\ \mathit{front}' = \mathit{front})\ \mathit{prophs}\ \textbf{with} \\
\quad |\ [] \Rightarrow \boxed{\bullet - .\ \circ -}^{\gamma.\mathrm{winner}} \\
\quad |\ (\_, \mathit{id}) :: \_ \Rightarrow \\
\qquad \bigvee\left[\begin{array}{l}
\boxed{\bullet - .\ \circ -}^{\gamma.\mathrm{winner}} \\
\text{identifier}\ \mathit{id} \ast \exists\, \Phi.\ \boxed{\bullet\,(\mathit{front}, \Phi)}^{\gamma.\mathrm{winner}} \ast \text{chaselev-au}\ \gamma\ \iota\ \Phi
\end{array}\right.
\end{array}\right.
$$

State 3 (emptyish)

$$\text{chaselev-state}_3\ \gamma\ \mathit{front}\ \mathit{back}\ \mathit{hist}\ \mathit{prophs} \triangleq$$

$$\ast \left[\begin{array}{l} \mathit{front} = \mathit{back} \\ \boxed{\bullet\, \mathit{hist}}^{\gamma.\mathrm{hist}} \ast |\mathit{hist}| = \mathit{front} + 1 \\ \quad \textbf{match}\ \mathrm{filter}\ (\lambda(\mathit{front}', \_).\ \mathit{front}' = \mathit{front})\ \mathit{prophs}\ \textbf{with} \\ \quad |\ [] \Rightarrow \boxed{\circ\,(\mathit{front}, -)}^{\gamma.\mathrm{winner}} \\ \quad |\ \_ \Rightarrow \exists\, \Phi.\ \boxed{\bullet\,(\mathit{front}, \Phi)}^{\gamma.\mathrm{winner}} \ast \Phi\ (\texttt{SOME}\ \mathit{hist}[\mathit{front}]) \end{array}\right.$$

State 4 (super empty)

$$\text{chaselev-state}_4\ \gamma\ \mathit{front}\ \mathit{back}\ \mathit{hist} \triangleq$$

$$\ast \left[\begin{array}{l} \mathit{front} = \mathit{back} + 1 \\ \boxed{\bullet\, \mathit{hist}}^{\gamma.\text{hist}} \ast |\mathit{hist}| = \mathit{front} \\ \boxed{\bullet\, - \,.\ \circ\, -}^{\gamma.\text{winner}} \end{array}\right.$$