Gradual Typing: A New Perspective

With polymorphism, unions, intersections, and much more

G. Castagna, **V. Lanvin**, T. Petrucciani, J. Siek

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Gradual Typing (1/3)

Let's write a map, that can work on both arrays and lists depending on a condition:

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Runtime checks or **casts** are then inserted **automatically** by the compiler.

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- How: by adding a **dynamic type**, denoted "?".
- **Allows for a trade-off between safety and programming productivity.**

The transition is **gradual**:

$$\texttt{?} \preccurlyeq \texttt{?} \to \texttt{?} \preccurlyeq \texttt{Int} \to \texttt{?} \preccurlyeq \texttt{Int} \to \texttt{Bool}$$

Gradual Typing (3/3)

Sometimes this **gradualization** is too **coarse**

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let map (condition : Bool) (f : α -> β) (data : ?) : ? =
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We want to give the programmer a way to reject such cases **statically**, while still **accepting this function**.

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Unfortunately, this is **not well-typed** without additional checks, since α `array` $\vee$ α `list` $\not\leq$ α `array`.

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This is **safer**, but **extremely verbose**.

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- In **Semantic subtyping**,

  Types $\simeq$ Sets of values

  Subtyping $\simeq$ Set-containment

Pros and Cons

Set-theoretic types	Gradual types
Safe	Unsafe
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Can we get the **best of both worlds?**

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– By **subtyping**, $(\alpha\ \texttt{list} \vee \alpha\ \texttt{array}) \wedge ? \leq ?$.

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2. Embed this relation into typing rules.

$$\frac{\Gamma \vdash e_1 : \tau_1 \to \tau_1' \qquad \Gamma \vdash e_2 : \tau_2 \qquad \tau_2 \mathrel{\widetilde{\leq}} \tau_1}{\Gamma \vdash e_1 \; e_2 : \tau_1'}$$

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This gets even more complicated with set-theoretic types!

Declarative Systems

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Main idea: interpret occurrences of `?` as arbitrary **type variables**.

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Important remark: this translation is **only used** to define and compute relations, and **is not done in the source program**.

Discrimination and Materialization

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And we define **materialization** (which is the inverse of **precision**, as defined in Garcia [2013]):

$$\tau_1 \preccurlyeq \tau_2 \overset{\text{def}}{\iff} \exists T_1 \in \mathcal{D}(\tau_1), \sigma : \mathtt{Vars} \to \mathtt{GTypes},\ T_1\sigma = \tau_2$$

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As well as **gradual subtyping**:

$$\tau_1 \leq \tau_2 \overset{\text{def}}{\iff} \exists (T_1, T_2) \in \mathcal{D}(\tau_1) \times \mathcal{D}(\tau_2), T_1 \leq_T T_2$$

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It can be used to handle unions and intersections, by **simply plugging-in** the static version of **semantic subtyping**:

$$? \leq ? \vee \mathsf{Int} \qquad \mathsf{Int} \wedge ? \leq ?$$

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Therefore it can be embedded into a type system as a **subsumption rule**.

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And as a bonus, we get the **static gradual guarantee** for free!

Back to the Example (1/2)

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$\Longrightarrow$ `Array.map f data` is **well-typed.**

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Now **from the outside**, consider a partial application `f`:

$$\Gamma \vdash \texttt{f} \;: ((\alpha\ \texttt{array} \vee \alpha\ \texttt{list}) \wedge ?) \to \texttt{t}$$

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Simply materializing $\texttt{?}$ **does not work**:

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Subtyping cannot be used either as it is **contravariant in the domain**:

$$((\alpha \; \texttt{array} \vee \alpha \; \texttt{list}) \wedge \texttt{?}) \to \texttt{t} \not\leq \texttt{?} \to \texttt{t}$$

Is This Still Gradual Typing?

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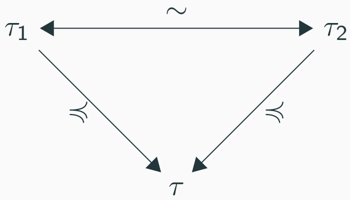

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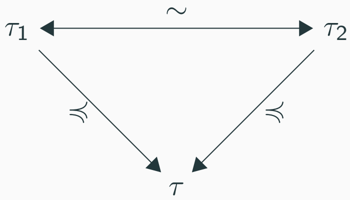

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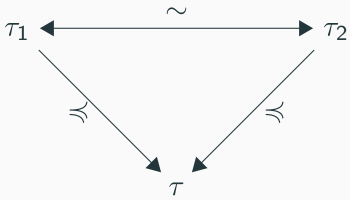

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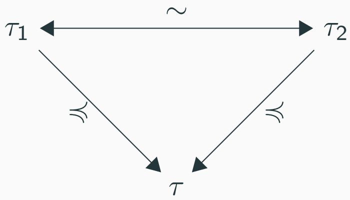

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3- Same results for the polymorphic system of Garcia & Cimini [2015].

Towards a Cast Language

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Blame tells us where an error occurred, and **in which way** the boundary was crossed.

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Moreover, the direction of the cast **can be enforced** in the typing rules:

$$\frac{\Gamma \vdash e : \tau_1 \qquad \begin{cases} p = I & \implies \tau_1 \preccurlyeq \tau_2 \\ p = \bar{I} & \implies \tau_2 \preccurlyeq \tau_1 \end{cases}}{\Gamma \vdash e \langle \tau_1 \overset{p}{\Rightarrow} \tau_2 \rangle : \tau_2}$$

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Blame safety is an important result of the cast language that states that **only the dynamically-typed part** of the code can cause errors.

We only insert casts when crossing from **dynamic** to **static** code, and **precisely control** the direction of each cast throughout the execution. **This makes proving blame safety straightforward.**

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– We presented a simple, straightforward way of **declaratively adding gradual typing** to existing type systems and compilation systems.

– We highlight a **direct correspondence** between compilation and type derivations.

– The declarative systems enjoy **many** (almost) **free theorems** (blame safety, type preservation, static gradual guarantee).

Algorithmic Systems

static types $\mathcal{T}_t \ni t ::= \alpha \mid b \mid t \times t \mid t \to t$

gradual types $\mathcal{T}_\tau \ni \tau ::= ? \mid \alpha \mid b \mid \tau \times \tau \mid \tau \to \tau$

source language $e ::= x \mid c \mid \lambda x.\, e \mid \lambda x : \tau.\, e \mid e\ e \mid (e, e) \mid \pi_i\ e$
$\mid \text{let}\ \vec{\alpha}\ x = e\ \text{in}\ e$

cast language $E ::= \lambda^{\tau \to \tau} x.\, E \mid \text{let}\ x = E\ \text{in}\ E \mid \Lambda \vec{\alpha}.\, E \mid E\, [\vec{t}]$
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Inference: Main Ideas

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Note: we **never infer gradual types**, they can only be introduced by **explicit annotations**.

Inference: Structured Constraints

We first **generate constraints** of the form[1]:

$$C ::= (t \dot{\leq} t) \mid (\tau \dot{\preccurlyeq} \alpha) \mid (x \dot{\preccurlyeq} \alpha) \mid \text{def } x : \tau \text{ in } C \mid \exists \vec{\alpha}.\, C \mid C \wedge C$$

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\end{aligned}$$

Note that $\langle\!\langle (\lambda x\colon ?.\, x)\colon \mathsf{Int} \to \mathsf{Int} \rangle\!\rangle$ **can be solved**, whereas $\langle\!\langle (\lambda x.\, x)\colon ? \to ? \rangle\!\rangle$ **cannot**.

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Rewriting constraints

We then **rewrite the structured constraints** to obtain a set containing **type constraints**:

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$$\frac{(\Gamma, x\colon \tau); \Delta \vdash C \rightsquigarrow D}{\Gamma; \Delta \vdash \text{def } x\colon \tau \text{ in } C \rightsquigarrow D}$$

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and solving it will return the unifier

$$\theta : X_1 \mapsto \mathsf{Bool}; X_2 \mapsto \beta; X_3 \mapsto \gamma; \alpha \mapsto (\beta \to \gamma)$$

Compilation and Results

To summarize, given an expression e, and a constraint derivation $\mathcal{D}$ of $\Gamma; \Delta \vdash \langle\!\langle e \colon t \rangle\!\rangle \rightsquigarrow D$, we can **compute a unifier** θ satisfying $\mathcal{D}$.

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Inference (and compilation) for this system is **sound**, **type-preserving** and **complete** w.r.t. the declarative system.

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For example,

```
fun x -> if (fst x) then (1 + snd x) else x
```

should be of type `(Bool`$\times$`Int)` $\rightarrow$ `( Int | (Bool`$\times$`Int) )`

The types become:

static types	$t ::= \alpha \mid b \mid t \times t \mid t \to t \mid t \vee t \mid \neg t \mid \mathbb{0}$
gradual types	$\tau ::= ? \mid \alpha \mid b \mid \tau \times \tau \mid \tau \to \tau \mid \tau \vee \tau \mid \neg \tau \mid \mathbb{0}$

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Soundness still holds for the inference algorithm, but **completeness no longer holds**.

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This is enough to decide subtyping

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Adding set-theoretic types is more complicated, mostly because we need to take into account unions and intersections containing ?.

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$$(\mathsf{Int} \to \mathsf{Int}) \vee (\mathsf{Bool} \to \mathsf{Bool})/_{(\mathsf{Int} \to \mathsf{Int}) \vee ?} = (\mathsf{Int} \to \mathsf{Int}) \vee (? \to ?)$$

$$[\text{ExpandL}] \quad V\langle \tau_1 \overset{p}{\Rightarrow} \tau_2 \rangle \quad \hookrightarrow \quad V\langle \tau_1 \overset{p}{\Rightarrow} {}^{\tau_1}/_{\tau_2} \rangle \langle {}^{\tau_1}/_{\tau_2} \overset{p}{\Rightarrow} \tau_2 \rangle$$

The full semantics are **conservative**, but complicated and contain **six additional rules to handle corner cases**.

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– We provided **sound semantics** for a cast calculus with set-theoretic gradual types and polymorphism.

The End

Thanks for listening!

Comments, questions, suggestions?