POPLMark Reloaded: Mechanizing Logical Relations Proofs

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beluga

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Mechanizing formal systems together with proofs establishes trust.

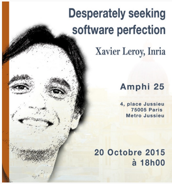
Mechanizing formal systems together with proofs establishes trust... and avoid flaws.

Programs go wrong.

Programs go wrong.

Testing correctness of C Compilers [Vu et.al PLDI'14]:

- GCC and LLVM had over 195 bugs
- Compcert the only compiler where no bugs were found



"This is a strong testimony to the promise and quality of verified compilers. "

[Vu et al PLDI'14]

Programming lang. designs and implementations go wrong.

Type Safety of Java (20 years ago)

Java is Type Safe — Probably

Sophia Drossopoulou and Susan Eisenbach

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Abstract. Amidst rocketing numbers of enthusiastic Java programmers and internet applet users, there is growing concern about the security of executing Java code produced by external, unknown sources. Rather than waiting to find out empirically what damage Java programs do, we aim to examine first the language and then the environment looking for points of weakness. A proof of the soundness of the Java type system is a first, necessary step towards demonstrating which Java programs won't compromise computer security.
We consider a type safe subset of Java describing primitive types, classes, inheritance, instance variables and methods, interfaces, shadowing, dynamic method binding, object creation, null and arrays. We argue that for this subset the type system is sound, by proving that program execution preserves the types, up to subclasses/subinterfaces.

Programming lang. designs and implementations go wrong.

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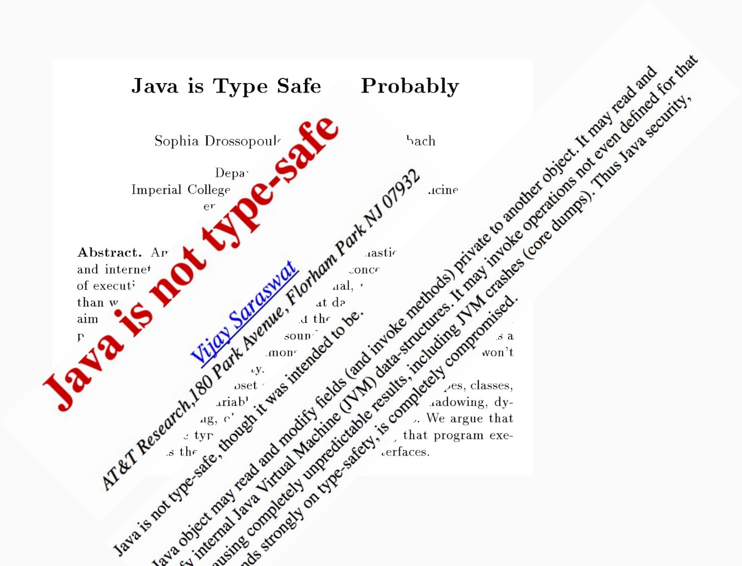
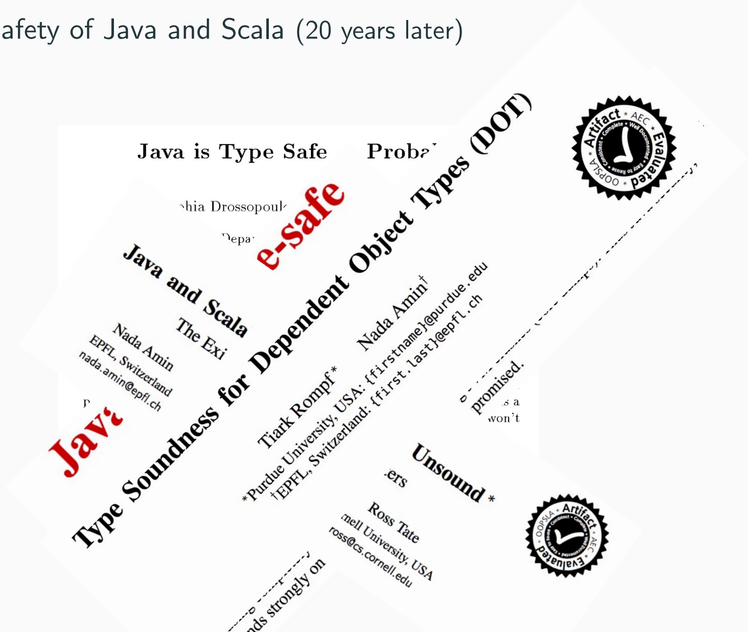


Type Safety of Java and Scala (20 years later)

Programming lang. designs and implementations go wrong.

Type Safety of Java and Scala (20 years later)



It’s a tricky business

“The truth of the matter is that putting languages together is a very tricky business. When one attempts to combine language concepts, unexpected and counterintuitive interactions arise. At this point, even the most experienced designers intuition must be butressed by a rigorous definition of what the language means. ”

- J. Reynolds

The problem: Correct proofs are tricky to write.

- a lot of overhead
  (on paper and even more so in a proof assistant)
- challenging to keep track of details
- hard to understand interaction between different features
- difficulties increase with size

What are good high-level proof languages that make it easier to mechanize and maintain formal guarantees?

POPLMark – A Look Back ...

POPLMark Challenge: Mechanize System $F_{<}$ [Aydemir et. al. 2005]

Spotlight on

"type preservation and soundness, unique decomposition properties of operational semantics, proofs of equivalence between algorithmic and declarative versions of type systems."

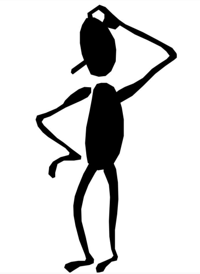
- Structural induction proofs (syntactic)
- Representing and reasoning about structures with binders
- Easy to be understood; text book description (TAPL)
- Small (can be mechanized in a couple of hours or days)
- Explore different encoding techniques for representing bindings

POPLMark Challenge – The Good

- ✓ Popularized the use of proof assistants
- ✓ Many submitted solutions
- ✓ Good way to learn about a technique
- ✓ Mechanizing proofs is addictive!

POPLMark Challenge – The Bad

- Did we achieve *"a future where the papers in conferences such as POPL and ICFP are routinely accompanied by mechanically checkable proofs of the theorems they claim."*?
- Did we get better tool support for mechanizing proofs?



POPLMark Challenge – The Ugly

- ✗ Did not identify bugs or flaws in existing systems
- ✗ Did not inspire the development of new theoretical foundations
- ✗ Did not push existing systems to their limit

> *"Type soundness results are two a penny."*
>
> Andrew Pitts

Beyond the POPLMark Challenge!

Beyond the POPLMark Challenge

"The POPLMark Challenge is not meant to be exhaustive: other aspects of programming language theory raise formalization difficulties that are interestingly different from the problems we have proposed - to name a few: more complex binding constructs such as mutually recursive definitions, logical relations proofs, coinductive simulation arguments, undecidability results, and linear handling of type environments." [Aydemir et. al. 2005]

POPLMark Reloaded – Goals and Target Audience

User Community Including Students

- Learn logical relations proofs a modern way
- Be able to grow the development to rich type theories (for example dependently typed systems)
- Understand the trade-offs in choosing a particular proof environment when tackling such a proof

Framework Developers

- Highlight features that are ideally suited for built-in support
- Highlight current shortcomings (theoretical and practical) in existing proof environments
- Signpost to advertise a given system
- Stimulate research on foundations of proof environments
- Benchmark for evaluating and comparing systems

POPLMark Reloaded:

Strong normalization for the simply-typed lambda-calculus with typed-reductions using Kripke-style logical relations

Simply Typed λ-Calculus with Type-Directed Reductions

Simply Typed λ-calculus:

$$
\begin{array}{llll}
\text{Terms} & M, N & ::= & x \mid \lambda x{:}A.M \mid M\ N \\
\text{Types} & A, B & ::= & A \Rightarrow B \mid \mathbf{i}
\end{array}
$$

Simply Typed λ-Calculus with Type-Directed Reductions

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Type-directed reductions [Goguen'95]: $\boxed{\Gamma \vdash M \longrightarrow N : A}$

$$\frac{\Gamma \vdash \lambda x{:}A.M : A \Rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash (\lambda x{:}A.M)\ N \longrightarrow [N/x]M : B}\ \beta$$

$$\frac{\Gamma \vdash M \longrightarrow M' : A \Rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash M\, N \longrightarrow M'\, N : B} \qquad \frac{\Gamma \vdash M : A \Rightarrow B \quad \Gamma \vdash N \longrightarrow N' : A}{\Gamma \vdash M\, N \longrightarrow M\, N' : B}$$

$$\frac{\Gamma, x{:}A \vdash M \longrightarrow M' : B}{\Gamma \vdash \lambda x{:}A.M \longrightarrow \lambda x{:}A.M' : A \Rightarrow B}$$

Why Type-directed Reductions?

- Simplifies the study of its meta-theory.
- Concise presentation of the important issues that arise.
- Widely applicable in studying subtyping, type-preserving compilation, etc.
- Types are necessary if we want η-expansion.

$$\frac{M \neq \lambda y{:}A.M'}{\Gamma \vdash M \longrightarrow \lambda x{:}A.M\ x : A \Rightarrow B} *$$

WARNING: This rule doesn't actually work, if we want strong normalization. Use at your own risk.

Normalization

- A term M is said to be *weakly normalising* if there is a rewrite sequence starting in M that eventually ends in a normal form.
- A term M is said to be *strongly normalising* if all rewrite sequences starting in M end eventually in a normal form.

Setting the Stage: How to Define Strong Normalization?

Often defined as an accessibility relation:

$$\frac{\forall N.\ \Gamma \vdash M \longrightarrow N : A \Longrightarrow \Gamma \vdash N : A \in \mathsf{sn}}{\Gamma \vdash M : A \in \mathsf{sn}}$$

> *"the reduct analysis becomes increasingly annoying in normalization proofs for more and more complex systems."*
> *Joachimski and Matthes [2003]*

Intuition ....How to Define Normalization

[F. van Raamsdonk and P. Severi 1995]

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How to obtain the set of all strongly normalising terms?

$\Longrightarrow$ Similar ... with a few restrictions

A Modular Approach to Strongly Normalizing Terms
[F. van Raamsdonk and P. Severi 1995]

- Inductive characterization of normal forms ($\Gamma \vdash M : A \in \mathsf{SN}$)
- Leads to modular proofs – on paper and in mechanizations

"the new proofs are essentially simpler than already existing ones." *F. van Raamsdonk and P. Severi*

Inductive definition of well-typed strongly normalizing terms

Neutral terms

$$\frac{x{:}A \in \Gamma}{\Gamma \vdash x : A \in \mathsf{SNe}} \qquad \frac{\Gamma \vdash R : A \Rightarrow B \in \mathsf{SNe} \quad \Gamma \vdash M : A \in \mathsf{SN}}{\Gamma \vdash R\, M : B \in \mathsf{SNe}}$$

Normal terms

$$\frac{\Gamma \vdash R : A \in \mathsf{SNe}}{\Gamma \vdash R : A \in \mathsf{SN}} \qquad \frac{\Gamma, x{:}A \vdash M : B \in \mathsf{SN}}{\Gamma \vdash \lambda x{:}A.M : A \Rightarrow B \in \mathsf{SN}}$$

$$\frac{\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : A \qquad \Gamma \vdash M' : A \in \mathsf{SN}}{\Gamma \vdash M : A \in \mathsf{SN}}$$

Strong head reduction

$$\frac{\Gamma \vdash N : A \in \mathsf{SN} \quad \Gamma, x{:}A \vdash M : B}{\Gamma \vdash (\lambda x.M)\; N \longrightarrow_{\mathsf{SN}} [N/x]M : B} \qquad \frac{\Gamma \vdash R \longrightarrow_{\mathsf{SN}} R' : A \Rightarrow B \quad \Gamma \vdash M : A}{\Gamma \vdash R\, M \longrightarrow_{\mathsf{SN}} R'\, M}$$

Challenge 1: Equivalence between accessibility and inductive definition of strongly normalizing terms:

$\Gamma \vdash M : A \in \mathsf{sn}$ **iff** $\Gamma \vdash M : A \in \mathsf{SN}$**.**

Strong Normalization using Logical Predicate

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Definition (Reducibility Candidates: $\Gamma \vdash M \in \mathcal{R}_A$)

$$
\begin{array}{lll}
\Gamma \vdash M \in \mathcal{R}_{\mathsf{i}} & \text{iff} & \Gamma \vdash M : \mathsf{i} \in \mathsf{SN} \\
\Gamma \vdash M \in \mathcal{R}_{A \Rightarrow B} & \text{iff} & \Gamma \vdash M : A \Rightarrow B \text{ and} \\
 & & \text{for all } N, \Delta \text{ such that } \Gamma \leq_\rho \Delta, \\
 & & \quad \text{if } \Delta \vdash N \in \mathcal{R}_A \text{ then } \Delta \vdash ([\rho]M)N \in \mathcal{R}_B.
\end{array}
$$

- Contexts arise naturally.
- They are necessary!
- The definition scales to dependently typed setting and stating properties about type-directed equivalence of lambda-terms.

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Do we really need the weakening substitution ρ?

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Do we really need to model terms in a "local" context and use Kripke-style context extensions?

Challenge 2: Strong normalization for simply typed λ-calculus

CR 1 : **If** $\Gamma \vdash M \in \mathcal{R}_A$ **then** $\Gamma \vdash M : A \in SN$**.**

CR 2 : **If** $\Gamma \vdash R : A \in \mathsf{SNe}$ **then** $\Gamma \vdash R \in \mathcal{R}_A$**.**

CR 3 : **If** $\Gamma \vdash M \longrightarrow_{\mathsf{SN}} M' : A$ **and** $\Gamma \vdash M' \in \mathcal{R}_A$ **then** $\Gamma \vdash M \in \mathcal{R}_A$**.**

Main fundamental lemma:

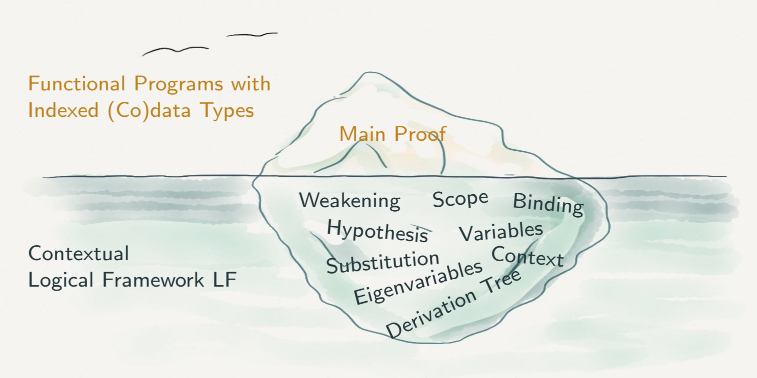
If $\Gamma \vdash M : A$ **and** $\Gamma' \vdash \sigma \in \mathcal{R}_\Gamma$ **then** $\Gamma' \vdash [\sigma]M \in \mathcal{R}_A$**.**

Challenges in the Proof(s)

- Definitions use well-typed terms
- Stratified definitions for reducibility candidates
  (not strictly positive!)
- Simultaneous substitutions and weakenings
- Basic infrastructure
  - Substitution properties about terms
  - Weakening and Strengthening of type-directed reductions
  - Weakening, Exchange, and Strengthening for typing
  - Weakening, Anti-weakening for strongly normalizing terms
  - Weakening for reducibility candidates
- Induction principles

Towards solving the challenge problems

Beluga: Programming and Proof Environment



- Below the surface: Support for key concepts based on Contextual LF [TOCL'08,POPL'08,LFMTP'13, ESOP'17, ...]
- Above the surface: (Co)Inductive Proofs
  as (Co)Recursive Programs using (Co)pattern Matching
  [POPL'08,IJCAR'10, POPL'12,POPL'13,CADE'15,ICFP'16, ...]

A Quick Guided Tour

Demo

Mechanization of Strong Normalization for STLC in Beluga

- Use HOAS to characterize simply typed terms
- Define SN inductively
- Use stratified definition for reducibility
- Extension to disjoint sums.

Lessons Learned – The Good, the Bad and the Ugly

- ✓ HOAS is great to model binding structure!
- ✓ Built-in support for substitutions and weakening is very useful!
- ✓ Take advantage of dependent types to model intrinsically typed terms, typed-reductions, typed SN, etc.
- ✓ Great to investigate and motivate extensions, first-class weakenings, to the theory of simultaneous substitutions
  Unification in the presence of renamings.
- ✓ Great to find bugs and make system more robust.
  Particularly coverage and termination checking
  Motivated lexicographic orderings for termination checking
  Ordering does not take into account exchange
- ✗ Interactive proof development mode clearly needs work.
- ✗ No proof automation – know what you want to do.

Other Mechanizations: Coq [S. Schäfer, K. Stark]

- Use well-typed de Bruijn encoding for simply typed terms

```
Inductive tm (Γ : ctx) : ty → Type :=
 | var A : Var Γ A → tm Γ A
 | app A B : tm Γ (A ⇒ B) → tm Γ A → tm Γ B
 | lam A B : tm (A :: Γ) B → tm Γ (A ⇒ B).
```

- Weakenings (renamings) are functions mapping positions in a context to positions in another context.

$$\rho\colon \forall\ \mathtt{A},\ \mathtt{Var}\ \Gamma\ \mathtt{A} \to \mathtt{Var}\ \Gamma'\ \mathtt{A}.$$

- Substitutions are functions mapping positions in a context to terms in another context.

$$\sigma\colon \forall\ \mathtt{A},\ \mathtt{Var}\ \Gamma\ \mathtt{A} \to \mathtt{tm}\ \Gamma'\ \mathtt{A}$$

- Following explicit substitution calc. [Abadi et al 1991]

Coq: The Good, the Bad, and the Ugly

- ✓ Everything could be proven "as expected"
- ✓ Repetitive proofs (renaminig, anti-renaming lemmas) proven using proof scrips and Coq tactics.
- ✗ Approx. 200 lines of technical boilerplate code

  Rethinking Autosubst: well-suited for automatically generating boilerplate code for well-scoped representations (not well-typed ones)

Other Mechanizations: Agda [G. Allais]

- Use generic-syntax library to define well-typed de Bruijn encoding for simply typed terms
  (Type $\times$ List Type)–indexed functor
  Library size: 1600 LOC
- Theory of renaming and substitution is not internalised.
- Worked well for expert user
- Lead to implementing new generic results as part of the generic-syntax library that were previously overlooked

Comparisons LOC (to be taken cautiously...)

	Beluga		Coq		Agda	
	STLC	**STLC+**	**STLC**	**STLC+**	**STLC**	**STLC+**
Library (generic-syntax)					1600	1600
Typed Terms & Red.	75	174	202	265	85	144
Def. Strong Norm.	30	48	22	30	34	47
Twist: Eval. Ctx					38	57
Properties of sn	163	355	97	214	97	219
Soundness of SN	29	50	17	24	12	18
Properties of SN	69	136	43	67	42	65
SN proof (Log. Pred.)	80	155	55	115	70	134
Total	446	918	446	715	378	684

Isn't proving strong normalization in a proof assistant an old hat?

Just Following Girard's "Proofs and Types" An Incomplete Bibliography

T. Altenkirch [TLCA'93]	: SN for System F
B. Barras and B. Werner [1997]	: SN for CoC
C. Coquand [1999]	: NbE for $\lambda\sigma$
S. Berghofer [TYPES 2004]	: WN for STL

Characteristic Features:

- Terms are not well-scoped or well-typed
- Candidate relation is untyped and does not enforce well-scoped terms

$\Longrightarrow$ does not scale to typed-directed evaluation or equivalence
$\Longrightarrow$ today we may have better techniques to structure proofs (inductive def. of SN)

Beyond strong normalization ...

Additional Challenge Problems

- Weak normalization
  (good starting point) [LFMTP'13,MSCS'17] ✓
- Type-directed algorithmic equality
  (Tutorial by K. Crary in ATPL; similar issues as in strong normalization with typed reductions) [LFMTP'15,MSCS'17] ✓
- Adding η-expansion ?
- Normalization of System F
  (excellent suggestion – relies on impredicativity) ✗

Let's systematically compare different mechanization.

Benchmarks can be great!

A Call for Action

- Choose your favorite proof assistant
- Complete the challenge
- Let's stick to the given set up
  in particular the inductive def. of SN ...
  - ✓ makes it easier to compare mechanization
  - ✓ it's good for you :-)

Let's get started... talk to me for the challenge problem set up and questions to keep in mind.

Thank you!

If you submit a solution, please answer the following set of questions to help us compare and evaluate different mechanizations.

Questions About the Set Up

- How are bindings, substitutions, and necessary infrastructre represented? How big is your initial set up? – If you use libraries, explain briefly what they are.
- How are well-typed terms modelled?
- How are Kripke-style context extensions modelled?
- How does the formalization deal with renaminigs / weakenings?

Questions About the Proof Development

- How does it compare to the proof given in the online tutorial?
- Were there any additional lemmas required besides the ones given in the tutorial?
- How straightforward was it to extend the language to unit and disjoint sums? Did anything in the set-up needed to be changed?

Questions About General Lessons

- Did you find solving the problem interesting? Did it expose you to a new perspective on logical relations proofs?
- Did solving this problem expose any issues with the system you were working with? Did it inspire extensions?
- Do you have any general lessons / take-aways?

Let's get started... We are looking forward seeing your solutions.