The PML$_2$ language:
Proving Programs in ML

Rodolphe Lepigre - Séminaire Gallium du 08/03/2018

Semantics and Implementation of an Extension of ML for Proving Programs

Université Savoie Mont Blanc

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A Programming Language, with Program Proving Features

An ML-like programming language with:

- records, variants (constructors), inductive types,
- polymorphism, general recursion,
- a call-by-value evaluation strategy,
- effects (control operators),
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- programs as individuals (higher-order layer),
- an equality type $t \equiv u$ (observational equivalence),
- a dependent function type (typed quantification).
- Termination checking is required for proofs.

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Example of Program and Proof

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type rec nat = [Zero ; S of nat]

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Part I

Specificities of the Type System

Examples of (equational) program properties:

- `add (add m n) k` ≡ `add m (add n k)` (associativity of `add`)
- `rev (rev l)` ≡ `l` (`rev` is an involution)
- `map g (map f l)` ≡ `map (fun x {g (f x)}) l` (`map` and composition)
- `sort (sort l)` ≡ `sort l` (`sort` is idempotent)

Properties as Program Equivalences

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Specification of a sorting function using predicates:

- `is_increasing (sort l)` $\equiv$ `true` (sort produces a sorted list)
- `is_perm (sort l) l` $\equiv$ `true` (sort yields a permutation)

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Remark: decision of equivalence only needs to be correct.

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We need a form of typed quantification!

Dependent Functions for Typed Quantification

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```
val rec add_n_Sm : ∀n m∈nat, add n S[m] ≡ S[add n m] =
  fun n m {
    case n { Zero → {} | S[k] → add_n_Sm k m }
  }

val rec add_comm : ∀n m∈nat, add n m ≡ add m n =
  fun n m {
    case n {
      Zero → add_n_Zero m
      S[k] → add_n_Sm m k; add_comm k m
    }
  }
```

Part II

Formalisation of the System and Semantics

We build a model to prove that the language has the expected properties.

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To construct the model, we need to:

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3) define adequate typing rules,
4) deduce *termination*, *type safety* and *consistency*.

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Advantage: it is modular (contrary to *type preservation*).

Values	(Λ_ι)	$v, w ::= x \mid \lambda x.t \mid \{(l_i = v_i)_{i\in I}\} \mid C_k[v]$	
Terms	(Λ)	$t, u ::= v \mid t\ u \mid v.l_k \mid [v \mid (C_i[x_i] \to t_i)_{i\in I}] \mid \mu\alpha.t \mid [\pi]t$	
Stacks	(Π)	$\pi, \xi ::= \alpha \mid \varepsilon \mid v \cdot \pi \mid [t]\pi$	(evaluation context)
Processes		$p, q ::= t * \pi$	

Call-by-value Reduction Relation

$$
\begin{aligned}
t\,u * \pi &\succ u * [t]\pi \\
v * [t]\pi &\succ t * v \,.\, \pi \\
\lambda x.t * v \,.\, \pi &\succ t[x := v] * \pi \\
\{(l_i = v_i)_{i \in I}\}.l_k * \pi &\succ v_k * \pi && (k \in I) \\
[C_k[v] \mid (C_i[x_i] \to t_i)_{i \in I}] * \pi &\succ t_k[x_k := v] * \pi && (k \in I) \\
\mu\alpha.t * \pi &\succ t[\alpha := \pi] * \pi \\
[\pi]t * \xi &\succ t * \pi
\end{aligned}
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Types as Sets of Canonical Values

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[\![\{l_1 : A_1 ; l_2 : A_2\}]\!] &= \left\{\{l_1 = v_1 ; l_2 = v_2\} \mid v_1 \in [\![A_1]\!] \ \wedge \ v_2 \in [\![A_2]\!]\right\} \\
[\![[C_1 : A_1 \mid C_2 : A_2]]\!] &= \left\{C_i[v] \mid i \in \{1, 2\} \ \wedge \ v \in [\![A_i]\!]\right\} \\
[\![\forall X.A]\!] &= \bigcap_{\Phi \text{ type}} [\![A[X := \Phi]]\!] \\
[\![\exists X.A]\!] &= \bigcup_{\Phi \text{ type}} [\![A[X := \Phi]]\!] \\
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We consider a new *membership type* $t \in A$ (with t a term, A a type).

- It is interpreted as $[\![t \in A]\!] = \{v \in [\![A]\!] \mid t \equiv v\}$,
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- and allows the introduction of dependency.

The dependent function type $\forall x \in A.B$

- is defined as $\forall x.(x \in A \Rightarrow B)$,
- this is a form of *relativised quantification* scheme.

We also consider a new *restriction type* $A \upharpoonright P$:

- it is build using a type A and a "semantic predicate" P,
- $[\![A \upharpoonright P]\!]$ is equal to $[\![A]\!]$ if P is satisfied and to $[\![\bot]\!]$ otherwise.
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The equality type $t \equiv u$ is encoded as $\top \upharpoonright t \equiv u$.

Interpretation of the Function Type

$$[\![A \Rightarrow B]\!] = \{\lambda x.w \mid \forall\, v \in [\![A]\!],\, w[x := v] \in [\![B]\!]\}$$

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[\![A]\!] &\in \{\Phi \subseteq \Lambda_\iota \mid v \in \Phi \wedge v \equiv w \Rightarrow w \in \Phi\} \\
[\![A]\!]^{\perp\!\!\!\perp} &= \{\pi \in \Pi \mid \forall v \in [\![A]\!], v * \pi \in \perp\!\!\!\perp\} \\
[\![A]\!]^{\perp\!\!\!\perp\perp\!\!\!\perp} &= \{t \in \Lambda \mid \forall \pi \in [\![A]\!]^{\perp\!\!\!\perp}, t * \pi \in \perp\!\!\!\perp\}
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Value Restriction and Typing Judgments

Combining call-by-value and effects leads to soundness issues (well-known).

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Theorem (adequacy lemma):

- if $\vdash t : A$ is derivable then $t \in [\![A]\!]^{\perp\!\!\!\perp\perp\!\!\!\perp}$;
- if $\vdash_{val} v : A$ is derivable then $v \in [\![A]\!]$.

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We only need to check that our typing rules are “correct”.

For example $\dfrac{\vdash_{\mathrm{val}} v : A}{\vdash v : A}$ is correct since $[\![A]\!] \subseteq [\![A]\!]^{\perp\!\!\perp\perp\!\!\perp}$.

Adequacy of For All Introduction

$$\frac{\Gamma;\ \Xi \vdash_{\mathrm{val}} v : A}{\Gamma;\ \Xi \vdash_{\mathrm{val}} v : \forall \mathsf{X}.A}\ \mathsf{X} \notin \Gamma$$

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$$\frac{X \vdash_{\mathrm{val}} v : A}{\vdash_{\mathrm{val}} v : \forall X.A}$$

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We suppose $t \in [\![A[X := \Phi]]\!]^{\perp\!\!\!\perp\perp\!\!\!\perp}$ for all Φ, and show $t \in [\![\forall X.A]\!]^{\perp\!\!\!\perp\perp\!\!\!\perp}$.

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However we have $\cap_\Phi [\![A[X := \Phi]]\!]^{\perp\!\!\!\perp\perp\!\!\!\perp} \not\subseteq [\![\forall X.A]\!]^{\perp\!\!\!\perp\perp\!\!\!\perp} = \left(\cap_\Phi [\![A[X := \Phi]]\!]\right)^{\perp\!\!\!\perp\perp\!\!\!\perp}$.

Theorem (normalisation):

$t : A$ implies $t * \varepsilon \succ v * \varepsilon$ for some value v.

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Theorem (safety for simple datatypes):

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Theorem (consistency):

there is no closed term $t : \bot$.

Part III

Semantical Value Restriction

Derived Rules for Dependent Functions

$$\frac{x : A \vdash t : B[a := x]}{\vdash_{val} \lambda x.t : \forall a \in A.B}$$

$$\frac{\vdash t : \forall a \in A.B \qquad \vdash_{val} v : A}{\vdash t\ v : B[a := v]}$$

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Value restriction breaks the compositionality of dependent functions.

```
// add_n_Zero : ∀n∈nat, add n Zero ≡ n
add_n_Zero (add Zero S[Zero]) : add (add Zero S[Zero]) Zero ≡ add Zero S[Zero]
```

We replace $\dfrac{\vdash t : \forall a \in A.B \qquad \vdash_{val} v : A}{\vdash t\ v : B[a := v]}$ by $\dfrac{\vdash t : \forall a \in A.B \qquad \vdash u : A \qquad \vdash u \equiv v}{\vdash t\ u : B[a := u]}$.

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Biorthogonal Completion Closed for Values

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The rule seems reasonable, but it is hard to justify semantically.

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The New Instruction Trick

We do not have $v \notin [\![A]\!]$ implies $v \notin [\![A]\!]^{\perp\!\!\!\perp\perp\!\!\!\perp}$ in every realizability model.

We extend the system with a new term constructor $\delta_{v,w}$ such that

$$\delta_{v,w} * \pi \succ v * \pi \qquad \text{iff} \qquad v \not\equiv w.$$

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- We can take $\pi = [\lambda x.\delta_{x,v}]\varepsilon$.
- $v * [\lambda x.\delta_{x,v}]\varepsilon \succ \lambda x.\delta_{x,v} * v\,.\,\varepsilon \succ \delta_{v,v} * \varepsilon \Uparrow$
- $w * [\lambda x.\delta_{x,v}]\varepsilon \succ \lambda x.\delta_{x,v} * w\,.\,\varepsilon \succ \delta_{w,v} * \varepsilon \succ w * \varepsilon \Downarrow$ if $w \in [\![A]\!]$

Well-defined construction of equivalence and reduction

Problem: the definitions of $(\succ)$ and $(\equiv)$ are circular.

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We need to rely on a stratified construction of the two relations.

$$(\twoheadrightarrow_i) \quad = \quad (\succ) \cup \left\{(\delta_{v,w} * \pi, v * \pi) \mid \exists j < i, v \not\equiv_j w\right\}$$

$$(\equiv_i) \quad = \quad \left\{(t, u) \mid \forall j \leqslant i, \forall \pi, \forall \sigma, t\sigma * \pi \Downarrow_j \Leftrightarrow u\sigma * \pi \Uparrow_j\right\}$$

We then take

$$(\twoheadrightarrow) \;=\; \bigcup_{i \in \mathbb{N}} (\twoheadrightarrow_i) \qquad \text{and} \qquad (\equiv) \;=\; \bigcap_{i \in \mathbb{N}} (\equiv_i).$$

Conclusion

Things That I did not Show

1) Syntax directed typing and subtyping rules using:
   - local subtyping judgments of the form $t \in A \subset B$,
   - choice operators like $\varepsilon_{x \in A}(t \notin B)$ or $\varepsilon_X(t \notin A)$,
   - an encoding of "neutral terms" into reduction.

2) Inductive types, coinductive types and recursion (more recent) using:
   - circular typing and subtyping proofs,
   - well-foundedness established using the *size change principle*.

3) Unreachable code and refutation of patterns.

Future Work

Practical issues (work in progress):

- Composing programs that are proved terminating.
- Extensible records and variant types (inference).

Toward a practical language:

- Compiler using typing informations for optimisations.
- Built-in types (int64, float) with their specification.

Theoretical questions:

- Can we handle more side-effects? (mutable cells, arrays)
- What can we realise with (variations of) $\delta_{v,w}$?
- Can we extend the system with quotient types?
- Can we formalise mathematics in the system?

References for Technical Details

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Thanks!