A Functional Synchronous Language with Time Warps

Adrien Guatto

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Streams in Programs and Proofs

- Infinite sequences of values

$$\mathsf{Stream}(X) \approx \mathbb{N} \to X$$

- Kahn's insight: a deterministic reactive system can be described as a mathematical function

$$\mathsf{Stream}(X) \to \mathsf{Stream}(Y)$$

- Exploited in various languages and formalisms:
  - lazy functional languages, e.g. Haskell;
  - synchronous dataflow languages, e.g. Lustre;
  - proof assistants based on Type Theory, e.g. Coq.

Recursive Stream Definitions

- Streams, as infinite objects, have to be introduced via self-referential definitions.
- For example, `zeroes` can be characterized as the solution of

$$\texttt{zeroes} = 0 :: \texttt{zeroes}$$

  and defined as such in Haskell, Lustre, and Coq.
- What about equations with several or no solutions?

$$\texttt{weird} = \texttt{weird}$$

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Demonstration 1

Try the above in our three prototypical languages.

Productivity and its Enforcement

Productivity

A stream definition is *productive* when any finite prefix of the stream can be computed in finite time.

Productivity can be enforced by:

- Syntactic criteria (e.g., Coq and Lustre)
  - ✓ Simple and well-understood
  - ✗ Anti-modular, inexpressive
- Type systems (e.g., guarded type theories, Lucid Synchrone)
  - ✓ Modular
  - ? Expressive

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Guarded Type Theories and the Later Modality

Nakano's Key Idea

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- Enrich the type language with a modality

$$\tau ::= \cdots \mid \blacktriangleright \tau$$

  and related operations.

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- Give appropriate types to the stream constructor/destructors;

$$(::) : X \to \blacktriangleright \mathsf{Stream}(X) \to \mathsf{Stream}(X)$$

$$\texttt{head} : \mathsf{Stream}(X) \to X \qquad \texttt{tail} : \mathsf{Stream}(X) \to \blacktriangleright \mathsf{Stream}(X)$$

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- Have a special typing rule for recursive definitions.

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"This expression has type $\blacktriangleright$ Stream(Int) but was expected to have type Stream(Int)."

Later and its Limitations

- Following Nakano, many works from Birkedal, Krishnaswami, McBride, Møgelberg, Bizjak and others, studying:
  - powerful (dependent) type systems;
  - denotational and operational semantics;
  - practical and theoretical use cases, from

$$\texttt{nat} = 0 :: \texttt{map}\ (\lambda \texttt{x}.\texttt{x} + 1)\ \texttt{nat}$$

    to step-indexed models of programming languages.

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- However, current guarded type theories struggle with...
  - mutual recursion:

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    $$\texttt{nat} = 0 :: \texttt{spos} \qquad \texttt{spos} = \texttt{map}\ (\lambda \mathrm{x}.\mathrm{x} + 1)\ \texttt{nat}$$

  - fine-grained dependencies:

    $$\begin{aligned} \texttt{thuemorse} = \ & \texttt{false} :: \texttt{tail}\ (\texttt{h}\ \texttt{thuemorse}) \\ & \texttt{where}\ \texttt{h}\ (\texttt{x} :: \texttt{xs}) = \texttt{x} :: (\texttt{not}\ \texttt{x}) :: \texttt{h}\ \texttt{xs} \end{aligned}$$

Beyond Later: Time Warps

- Models of guarded recursion interpret types by ω-indexed families of sets of observations.

$$(\mathsf{Stream}(\mathsf{Int}))_n \approx \mathsf{Int}^n$$

- The later modality applies a simple transformation to a type: delaying what can be observed one step into the future.

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Pulsar is a prototype implementation of these ideas.

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1 Introduction

2 Programming in a Language with Time Warps

3 Metatheoretical Aspects

4 Algorithmic Type Checking

5 Perspectives

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Overview

- **Pulsar** is based on the simply-typed λ-calculus extended with a built-in stream type.

$$\begin{aligned} \tau &::= \ \nu \mid \mathsf{Stream}(\tau) \mid \tau \to \tau \mid \tau \times \tau \mid \dots \\ \nu &::= \ \mathsf{Int} \mid \mathsf{Bool} \mid \mathsf{Char} \end{aligned}$$

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \qquad \frac{\Gamma, x : \tau_1 \vdash t : \tau_2}{\Gamma \vdash \mathtt{fun}\,(x : \tau_1).t : \tau_2} \qquad \frac{\Gamma \vdash t_1 : \tau_1 \to \tau_2 \qquad \Gamma \vdash t_2 : \tau_1}{\Gamma \vdash t_1\ t_2 : \tau_2}$$

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- To the above, it adds the warp modality

$$\tau ::= \cdots \mid {\color{red} *_p\, \tau}$$

plus guarded recursion, subtyping, and a new construct.

Time Warps

- Formally, warps are sup-preserving functions from $\omega + 1$ to itself, i.e. monotonic functions such that

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- We restrict ourselves to warps defined as running sums $\mathcal{O}\, p$ of ultimately periodic number sequences p.

$$(\mathcal{O}\, p)(i) = \sum_{j=0}^{j<i} p[j] \text{ for } 0 < i < \omega$$

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$$(\mathcal{O}\,(2))(i) = 2i \qquad (\mathcal{O}\,(\omega))(i) = \omega \text{ for } i > 0$$

Guarded Recursion in **Pulsar**

Demonstration 2

Let us try to write `zeroes`.

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Guarded recursion is formulated with $\blacktriangleright \tau \triangleq *_{0\,(1)}\,\tau$ as expected.

$$\frac{\Gamma, x : *_{0\,(1)}\,\tau \vdash e : \tau}{\Gamma \vdash \texttt{rec}\ (x : \tau).e : \tau}$$

Similarly, primitives have types

$$(::) : \tau \to *_{0\,(1)}\,\mathsf{Stream}(\tau) \to \mathsf{Stream}(\tau)$$

$$\texttt{head} : \mathsf{Stream}(\tau) \to \tau \qquad \texttt{tail} : \mathsf{Stream}(\tau) \to *_{0\,(1)}\,\mathsf{Stream}(\tau)$$

Warp Composition

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However, $\vdash \texttt{weird} : \ast_{(0)} \mathrm{Stream}(\tau)$ holds for any τ. Why?

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$$\frac{}{\ast_{p \ast q}\,\tau \equiv \ast_p \ast_q \tau}$$

The operator $\ast$, called *warp composition*, is characterized by

$$\mathcal{O}\,(p \ast q) = \mathcal{O}\,q \circ \mathcal{O}\,p$$

hence we have

$$\ast_{0\,(1)} \ast_{(0)} \mathsf{Stream}(\tau) \equiv \ast_{0\,(1) \ast (0)} \mathsf{Stream}(\tau) \equiv \ast_{(0)} \mathsf{Stream}(\tau)$$

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$$\begin{aligned}
\mathtt{map} &: \ *_{0\,(1)}\,((\mathsf{Int} \to \mathsf{Int}) \to \mathsf{Stream}(\mathsf{Int}) \to \mathsf{Stream}(\mathsf{Int})) \\
\mathtt{f} &: \ \mathsf{Int} \to \mathsf{Int} \\
\mathtt{xs} &: \ *_{0\,(1)}\,\mathsf{Stream}(\mathsf{Int})
\end{aligned}$$

Warping and Delays

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Let us try to write map.

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$$\frac{\Gamma \vdash e : \tau}{*_p \Gamma \vdash e \texttt{ by } p : *_p \tau} \qquad \frac{}{\tau \equiv *_{(1)} \tau} \qquad \frac{q \leq p}{*_p \tau <: *_q \tau}$$

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\texttt{xs} :&\quad *_{0\,(1)} \mathsf{Stream}(\mathsf{Int})
\end{aligned}$$

Putting it all Together: Mutual Recursion

Demonstration 4

Let us write `nat` and `spos`.

Putting it all Together: Fine-Grained Dependencies

Demonstration 5

Let us write `thuemorse`.

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A Two-Level Calculus

- Implicit terms correspond to user programs:

$$t ::= x \mid \texttt{fun}\,(x : \tau).t \mid t\ t \mid (t, t) \mid \texttt{pr}_{i\in\{0,1\}} t \mid \texttt{rec}\ (x : \tau).t \mid t\ \texttt{by}\ p$$

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- Explicit terms have coercions and syntax-directed typing rules:

$$\begin{aligned} e ::= {} & x \mid \texttt{fun}\,(x:\tau).e \mid e\ e \mid (e,e) \mid \texttt{pr}_{i\in\{0,1\}}e \mid \texttt{rec}\ (x:\tau).e \mid e\ \texttt{by}\ p \\ \mid {} & (t;\alpha) \mid (\gamma;t) \end{aligned}$$

$$\ldots \qquad \frac{\Gamma \vdash e : \tau \qquad \alpha : \tau <: \tau'}{\Gamma' \vdash (\alpha; e) : \tau'} \qquad \frac{\gamma : \Gamma' <: \Gamma \qquad \Gamma \vdash e : \tau}{\Gamma' \vdash (\gamma; e) : \tau}$$

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- Every explicit term e erases to a unique implicit term $\mathbf{U}(e)$.

The Dynamics of **Pulsar**

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- Denotational, as an interpretation in the *topos of trees*

$$[\![\tau]\!] \in |\widehat{\omega}| \qquad\qquad [\![\Gamma \vdash e : \tau]\!] \in \widehat{\omega}([\![\Gamma]\!], [\![\tau]\!])$$

Operational Semantics

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$$\frac{}{\texttt{nil} : \tau \;\texttt{@}\; 0} \qquad \frac{v_1 : \tau \;\texttt{@}\; n+1 \qquad v_2 : \mathsf{Stream}(\tau) \;\texttt{@}\; n}{v_1 :: v_2 : \mathsf{Stream}(\tau) \;\texttt{@}\; n+1} \qquad \cdots \qquad \frac{v : \tau \;\texttt{@}\; p(n)}{(p, v) : *_p \tau \;\texttt{@}\; n}$$

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$$\boxed{e; \sigma \Downarrow_n v}$$

$$\frac{}{e; \sigma \Downarrow_0 \texttt{nil}} \qquad \cdots \qquad \frac{e; \pi_2(\sigma) \Downarrow_{p(n)} v}{e \ \texttt{by} \ p; \sigma \Downarrow_n (p, v)} \qquad \frac{x.e; \sigma; \texttt{nil} \Uparrow_0^n v}{\texttt{rec} \ (x : \tau).e; \sigma \Downarrow_n v}$$

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$$\boxed{x.e; \sigma; v \Uparrow_m^n v'}$$

$$\frac{m < n \qquad e; \lfloor \sigma \rfloor_m [x \mapsto v] \Downarrow_{v'} \qquad x.e; \sigma; v' \Uparrow_{m+1}^n v''}{x.e; \sigma; v \Uparrow_m^n v''} \qquad \frac{m \geq n}{x.e; \sigma; v \Uparrow_m^n v}$$

Denotational Semantics: the Topos of Trees

Definition:

$$\widehat{\omega} \quad \triangleq \quad [\omega^{op}, \mathbf{Set}]$$

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$$X(0) \xleftarrow{r_0^X} X(1) \xleftarrow{r_1^X} X(2) \xleftarrow{r_2^X} X(3) \xleftarrow{r_3^X} X(4) \qquad \ldots$$

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$$\begin{array}{c} X \\ \downarrow f \\ Y \end{array} \qquad \begin{array}{ccccccccccc}
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Denotational Semantics: the Interpretation, Abridged

The language is mostly interpreted by exploiting the cartesian-closed structure of toposes, following Birkedal et al. In addition:

- Warps are (isomorphic to) endofunctors of ω, and thus of ω^{op}.
- Thus, if X is a presheaf, so is $X \circ p$. In other words:

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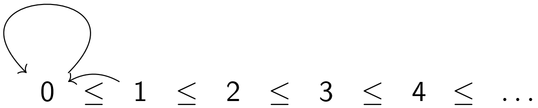

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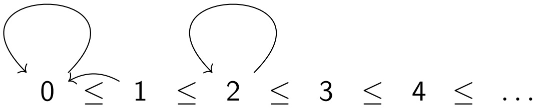

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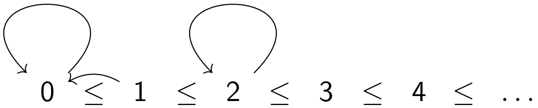

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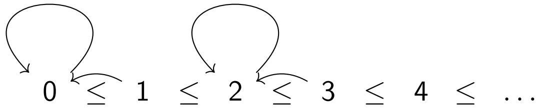

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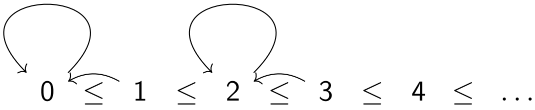

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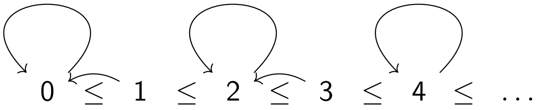

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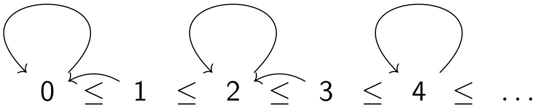

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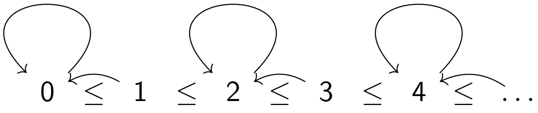

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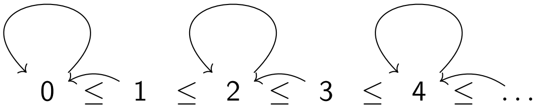

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Conjectural Results

Operational Semantics: Soundness and Totality

If $\Gamma \vdash e : \tau$ and $\sigma : \Gamma \mathbin{@} n$, then there is v s.t. $e; \sigma \Downarrow_n v$ and $v : \tau \mathbin{@} n$.

Denotational Semantics: Adequacy

If $[\![\Gamma \vdash e : \tau]\!] = [\![\Gamma \vdash e' : \tau]\!]$ then $\Gamma \vdash e \cong_{\mathrm{ctx}} e' : \tau$.

Outline

1 Introduction

2 Programming in a Language with Time Warps

3 Metatheoretical Aspects

4 Algorithmic Type Checking

5 Perspectives

Subtyping and Coherence

In `map`, we used

$$\mathtt{f} : \mathsf{Int} \to \mathsf{Int} \equiv *_{(1)}(\mathsf{Int} \to \mathsf{Int}) <: *_{0\,(1)}(\mathsf{Int} \to \mathsf{Int})$$

In fact, the compiler did

$$\begin{aligned}\mathtt{f} :\ \mathsf{Int} \to \mathsf{Int} \equiv *_{(1)}(\mathsf{Int} \to \mathsf{Int}) &<: *_{0\,2\,(1)}(\mathsf{Int} \to \mathsf{Int}) \\ &= *_{0\,(1)\,*\,2\,(1)}(\mathsf{Int} \to \mathsf{Int}) \\ &\equiv *_{0\,(1)}\,*_{2\,(1)}(\mathsf{Int} \to \mathsf{Int})\end{aligned}$$

and then

$$*_{2\,(1)}(\mathsf{Int} \to \mathsf{Int}) <: *_{(1)}(\mathsf{Int} \to \mathsf{Int}) <: \mathsf{Int} \to \mathsf{Int}$$

Coherence Issues

- Distinct explicit terms mean different things *a priori*.
- Programmer writes implicit terms, compiler elaborates them into explicit ones. Is this reasonable?

Type-Checking and Elaboration

Goals

- Define an algorithm $\Gamma \vdash t \rightsquigarrow e : \tau$ taking (Γ, t) and returning (e, τ) such that $\mathbf{U}(e) = t$ and $\Gamma \vdash e : \tau$.
- Make this choice canonical in a certain sense.

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In other words, we have to decide where to add $(-; \alpha)$ and $(\gamma; -)$ in t. This involves two main questions:

- Infer coercions $\alpha : \tau_1 <: \tau_2$ given τ_1 and τ_2.
- Infer coercions $\gamma : \Gamma_1 <: *_p \Gamma_2$ given Γ_1 and p.

Deciding Subtyping in Three Steps

1. The algorithmic judgment $\boxed{\tau \gg \tau^s \leadsto \alpha \rightleftarrows \alpha'}$:
   - implies $\alpha : \tau <: \tau^s$ and $\alpha' : \tau_s <: \tau$;
   - implies that τ^s respects the following grammar.

$$\begin{aligned} \tau^s &::= \ *_p\, \tau^r \mid \tau^s \times \tau^s \\ \tau^r &::= \ \nu \mid \mathsf{Stream}(\tau^s) \mid \tau^s \to \tau^s \end{aligned}$$

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   - holds iff τ is coercible to τ' using only delays.
3. Algorithmic subtyping $\boxed{\tau_1 <: \tau_2 \rightsquigarrow \alpha}$ can then be defined by

$$\frac{\tau_1 \gg \tau_1^s \rightsquigarrow \alpha_1 \rightleftarrows - \qquad \tau_2 \gg \tau_2^s \rightsquigarrow - \rightleftarrows \alpha_3 \qquad \tau_1^s \geq \tau_2^s \rightsquigarrow \alpha_2}{\tau_1 <: \tau_2 \rightsquigarrow \alpha_1; \alpha_2; \alpha_3}$$

Type-Checking Warping

To type-check e `by` p in Γ, for any τ in Γ we must find τ^p such that

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It reduces to a similar operation on warps.

$$q \geq p * r \Leftrightarrow q/p \geq r$$

Warp Division

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$$(\mathrm{Ran}_g(f))(n) = \prod f^*(g_*(\uparrow n))$$

The right Kan extension is presentable by a ultimately periodic sequence when both f and g are, and can be computed.

$$(1)/0\,(1) = 2\,(1) \qquad (1)/(2) = (1\,0) \qquad (1\,0)/(1\,0) = (1)$$

$$(1)/(0) = (\omega) \qquad (1)/(\omega) = 1\,(0)$$

Main Results

Coherence of Subtyping

If $\alpha : \tau_1 <: \tau_2$ and $\alpha' : \tau_1 <: \tau_2$ then

$$[\![\alpha : \tau_1 <: \tau_2]\!] = [\![\alpha' : \tau_1 <: \tau_2]\!]$$

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Completeness of Type-Checking

For any $\Gamma \vdash e : \tau$, there is e_m, τ_m, α such that

$$\Gamma \vdash \mathbf{U}(e) \rightsquigarrow e_m : \tau_m \qquad \Gamma \vdash e_m : \tau_m \qquad \alpha : \tau_m <: \tau$$

$$[\![\Gamma \vdash (e_m; \alpha) : \tau]\!] = [\![\Gamma \vdash e : \tau]\!]$$

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Corollary: Coherence

If $\Gamma \vdash e_1 : \tau$ and $\Gamma \vdash e_2 : \tau$ then

$$[\![\Gamma \vdash e_1 : \tau]\!] = [\![\Gamma \vdash e_2 : \tau]\!]$$

Outline

1 Introduction

2 Programming in a Language with Time Warps

3 Metatheoretical Aspects

4 Algorithmic Type Checking

5 Perspectives

Future Work

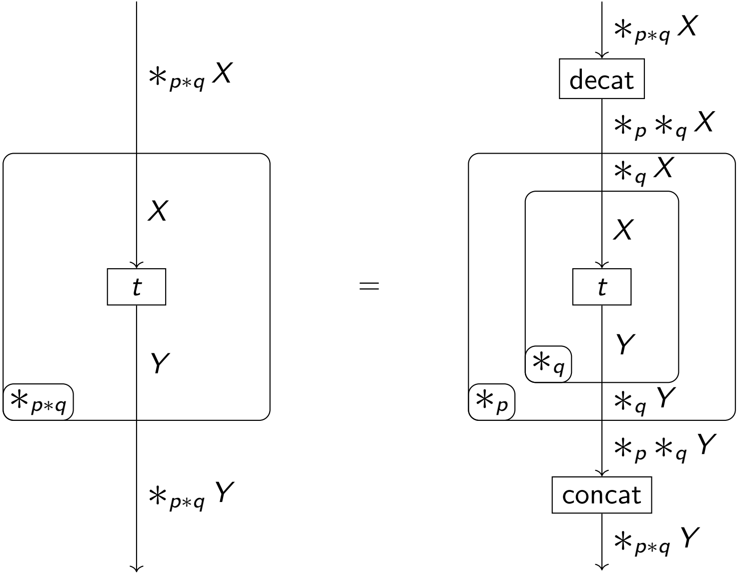
Frontend

- More ambitious subtyping.
- Type inference.

Backend

- Single-loop code generation.
- Typing restrictions to run within finite space.
-

What I Didn't Talk About



Conclusion

- I have presented a higher-order language with a rich notion of time. It handles programs that were previously out of reach of both synchronous languages and guarded type theories.
- Certain aspects of synchronous dataflow languages can be generalized through semantical intuitions in a natural way. I believe that this approach could be pushed much further.

Thank you!