MPRI 2.4, Functional programming and type systems
Metatheory of System F

Didier Rémy

Plan of the course

Simply typed lambda-calculus

Metatheory of System F

ADTs, Recursive types, Existential types, GATDs

Going higher order with F^{ω}!

Logical relations

Side effects, References, Value restriction

Type reconstruction

Overloading

Abstract Data types, Existential types, GADTs

Contents

- Algebraic Data Types
  - Equi- and iso- recursive types

- Existential types
  - Implicitly-type existential types passing
  - Iso-existential types

- Generalized Algebraic Datatypes

- Application to typed closure conversion
  - Environment passing
  - Closure passing

Algebraic Datatypes Types   Examples

In OCaml:

```
type 'a list =
 | Nil : 'a list
 | Cons : 'a * 'a list → 'a list
```

or

```
type ('leaf, 'node) tree =
 | Leaf : 'leaf → ('leaf, 'node) tree
 | Node : ('leaf, 'node) tree * 'node * ('leaf, 'node) tree → ('leaf, 'node) tree
```

Algebraic Datatypes Types   General case

General case

$$\text{type } G\,\vec{\alpha} = \Sigma_{i \in 1..n}(C_i : \forall \vec{\alpha}.\, \tau_i \to G\,\vec{\alpha}) \qquad \text{where } \vec{\alpha} = \bigcup_{i \in 1..n} \text{ftv}(\tau_i)$$

In System F, this amounts to declaring:

$$?$$

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- one destructor $\qquad d_G \;:\;\; \forall \vec{\alpha}, \gamma.\, G\,\vec{\alpha} \to (\tau_1 \to \gamma) \ldots (\tau_n \to \gamma) \to \gamma$

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- one destructor $d_G \;:\; \forall \vec{\alpha}, \gamma.\, G\,\vec{\alpha} \to (\tau_1 \to \gamma) \ldots (\tau_n \to \gamma) \to \gamma$
- n reduction rules $d_G\,\bar{\tau}\,(C_i\,\bar{\tau}'\,v)\,v_1\,\ldots v_n \;\rightsquigarrow\; v_i\,v$

Exercise
Show that this extension verifies the subject reduction and progress axioms for constants.

Algebraic Datatypes Types

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Notice that

- All constructors build values of the same type $G \, \vec{\alpha}$ and are surjective (all types can be reached)
- The definition may be recursive, *i.e.* G may appear in τ_i

Algebraic datatypes introduce *isorecursive types*.

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  - Equi- and iso- recursive types

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Recursive Types

Product and sum types alone do not allow describing *data structures* of *unbounded size*, such as lists and trees.

Indeed, if the grammar of types is $\tau ::= unit \mid \tau \times \tau \mid \tau + \tau$, then it is clear that every type describes a *finite* set of values.

For every k, the type of lists of length at most k is expressible using this grammar. However, the type of lists of unbounded length is not.

Equi- versus isorecursive types

The following definition is inherently *recursive:*

> *"A list is either empty or a pair of an element and a list."*

We need something like this:

$$list\ \alpha \quad \diamond \quad unit + \alpha \times list\ \alpha$$

But what does $\diamond$ stand for? Is it *equality,* or some kind of *isomorphism?*

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There are two standard approaches to recursive types:

- *equirecursive* approach:
    a recursive type is *equal* to its unfolding.

- *isorecursive* approach:
    a recursive type and its unfolding are related via explicit *coercions.*

Equirecursive types

In the equirecursive approach, the usual syntax of types:

$$\tau ::= \alpha \mid \mathsf{F}\ \vec{\tau} \mid \forall \beta.\, \tau$$

is no longer interpreted inductively. Instead, types are the *regular infinite trees* built on top of this grammar.

Finite syntax for recursive types

$$\tau ::= \alpha \mid \mu\alpha.(\mathsf{F}\ \vec{\tau}) \mid \mu\alpha.(\forall \beta.\, \tau)$$

We do not allow the seemingly more general form $\mu\alpha.\tau$, because $\mu\alpha.\alpha$ is meaningless, and $\mu\alpha.\beta$ or $\mu\alpha.\mu\beta.\tau$ are useless. If we write $\mu\alpha.\tau$, it should be understood that τ is contractive, that is, τ is a type constructor application or a forall introduction.

For instance, the type of lists of elements of type α is:

$$\mu\beta.(unit + \alpha \times \beta)$$

Equirecursive types   Equality

Inductive definition [Brandt and Henglein, 1998] show that equality is
the least congruence generated by the following two rules:

FOLD/UNFOLD
$\mu\alpha.\tau = [\alpha \mapsto \mu\alpha.\tau]\tau$

$$\frac{\text{UNIQUENESS}}{\tau_1 = [\alpha \mapsto \tau_1]\tau \qquad \tau_2 = [\alpha \mapsto \tau_2]\tau}{\tau_1 = \tau_2}$$

In both rules, τ must be contractive.

This axiomatization does not directly lead to an efficient algorithm for
deciding equality, though.

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\mu\alpha.\tau = [\alpha \mapsto \mu\alpha.\tau]\tau
\end{array}
\qquad
\begin{array}{c}
\text{Uniqueness} \\
\dfrac{\tau_1 = [\alpha \mapsto \tau_1]\tau \qquad \tau_2 = [\alpha \mapsto \tau_2]\tau}{\tau_1 = \tau_2}
\end{array}
$$

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Co-inductive definition

$$
\alpha = \alpha
\qquad
\dfrac{[\alpha \mapsto \mu\alpha.\mathsf{F}\vec{\tau}]\vec{\tau} = [\alpha \mapsto \mu\alpha.\mathsf{F}\vec{\tau}']\vec{\tau}'}{\mu\alpha.\mathsf{F}\vec{\tau} = \mu\alpha.\mathsf{F}\vec{\tau}'}
\qquad
\dfrac{[\alpha \mapsto \mu\alpha.\forall\beta.\tau]\tau = [\alpha \mapsto \mu\alpha.\forall\beta.\tau']\tau'}{\mu\alpha.\forall\beta.\tau = \mu\alpha.\forall\beta.\tau'}
$$

Equirecursive types   Equality

In the absence of quantifiers

Each type in this syntax denotes a unique regular tree, sometimes known as its *infinite unfolding.* Conversely, every regular tree can be expressed in this notation (possibly in more than one way).

If one builds a type-checker on top of this finite syntax, then one must be able to *decide* whether two types are *equal,* that is, have identical infinite unfoldings.

This can be done efficiently, either via the algorithm for comparing two DFAs, or better, by unification. (The latter approach is simpler, faster, and extends to the type inference problem.)

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Exercise
Show that $\mu\alpha.A\alpha = \mu\alpha.AA\alpha$ and $\mu\alpha.AB\alpha = A\mu\alpha.BA\alpha$ with both inductive and co-inductive definitions. Can you do it without the Uniqueness *rule?*

Equirecursive types   Without quantifiers

Proof of $\mu\alpha AA\alpha = \mu\alpha AAA\alpha$

By coinduction

Let $\Big|$ u be $\mu\alpha AA\alpha$
   v be $\mu\alpha AAA\alpha$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{(\mathbf{1})}{Au = Av}}{u = AAv}}{Au = v}}{u = Av}}{\cfrac{Au = AAv}{u = v \ (1)}}$$

Equirecursive types   Without quantifiers

Proof of $\mu\alpha AA\alpha = \mu\alpha AAA\alpha$

By coinduction

By unification

Equivalent classes, using *small terms*	To do:
$u \sim Au_1 \;\wedge\; u_1 \sim Au \;\wedge\; v \sim Av_1 \;\wedge\; v_1 \sim Av_2 \;\wedge\; v_2 \sim Av$	$u \sim v$
$u \sim Au_1 \sim v \sim Av_1 \;\wedge\; u_1 \sim Au \;\wedge\; v_1 \sim Av_2 \;\wedge\; v_2 \sim Av$	$u_1 \sim v_1$
$u \sim v \sim Av_1 \;\wedge\; u_1 \sim Au \sim v_1 \sim Av_2 \;\wedge\; v_2 \sim Av$	$u \sim v_2$
$u \sim v \sim Av_1 \sim v_2 \sim Av \;\wedge\; u_1 \sim v_1 \sim Av_2$	$v_1 \sim v$
$u \sim v \sim v_2 \sim Av \sim u_1 \sim v_1 \sim Av_2$	$v = v_2$
$u \sim v \sim v_2 \sim Av \sim u_1 \sim v_1 \sim Av_2$	$\varnothing$

Equirecursive types   Equality

In the presence of quantifiers

The situation is more subtle because of α-conversion.

A (somewhat involved) canonical form can still be found, so that checking equality and first-order unification on types can still be done in $O(n \log n)$. See [Gauthier and Pottier, 2004].

Otherwise, without the use of such canonical forms, the best known algorithm is in $O(n^2)$ [Glew, 2002] testing equality of automatons with binders.

Equirecursive types   With quantifiers

Example of unfolding with canonical forms [Gauthier and Pottier, 2004].

- the letter in green, is just any name, subject to α-conversion
- the number is the canonical name: it is the number of free variables under the binder—including recursive occurrences.

$$\forall a1. \mu\ell.a1 \to \forall a2. (a2 \to \ell) \tag{1}$$
$$\forall a1. \mu\ell.a1 \to \forall b2. (b2 \to \ell) \tag{α}$$
$$= \forall a1. \quad a1 \to \forall b2. (b2 \to \mu\ell.a1 \to \forall b2. (b2 \to \ell)) \tag{μ}$$
$$= \forall a1. \quad a1 \to \forall b2. (b2 \to \mu\ell.a1 \to \forall c2. (c2 \to \ell)) \tag{α}$$

With the canonical representation,

- Syntactic unfolding (*i.e.* without any renaming) avoids name capture and is also a correct semantic unfolding
- It shares free variables and can reuse the same name for the new bound variables without name capture.

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Equirecursive types   Type soundness

In the presence of equirecursive types, structural induction on types is no longer permitted, but *we never used it* anyway – in soundness proofs.

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Equirecursive types   Type soundness

In the presence of equirecursive types, structural induction on types is no
longer permitted, but *we never used it* anyway – in soundness proofs.

*We only need it to prove the termination of reduction, which does not
hold any longer.*

It remains true that

- $F\ \vec{\tau}_1 = F\ \vec{\tau}_2$ implies $\vec{\tau}_1 = \vec{\tau}_2$ (symbols are injective)—this is used in
  the proof of Subject Reduction.
- $F_1\ \vec{\tau}_1 = F_2\ \vec{\tau}_2$ implies $F_1 = F_2$—this is used in the proof of Progress.

So, the reasoning that leads to *type soundness* is unaffected.

Exercise
*Prove type soundness for the simply-typed λ-calculus in Coq. Then,
change the syntax of types from* Inductive *to* CoInductive.

Equirecursive types   break termination, indeed!

That is no a surprise, but...

What is the expressiveness of simply-typed λ-calculus with equirecursive types alone (no other constructs and/or constants)?

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What is the expressiveness of simply-typed λ-calculus with equirecursive types alone (no other constructs and/or constants)?

All terms of the untyped λ-calculus are typable!

- define the universal type U as $\mu\alpha.\alpha \rightarrow \alpha$
- we have $U = U \rightarrow U$, hence all terms are typable with type U.

Notce that one can emulate recursive types $U = U \rightarrow U$ by defining two functions *fold* and *unfold* of respective types $(U \rightarrow U) \rightarrow U$ and $U \rightarrow (U \rightarrow U)$ with side effects, such as:

- references, or
- exceptions

Equirecursive types   in OCaml

OCaml has both isorecursive and equirecursive types.

- equirecursive types are restricted by default to objects or datatypes.
- unrestricted equirecursive types are available upon explicit request.

Quiz: why so?

Isorecursive types

The folding/unfolding is witnessed by an explicit coercion.

The uniqueness rule is often omitted

(hence, the equality relation is weaker)

Encoding isorecursive types with ADT

The recursive type $\mu\beta.\tau$ can be represented in System F by introducing a datatype with a unique constructor:

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The recursive type $\mu\beta.\tau$ can be represented in System F by introducing a datatype with a unique constructor:

$$\text{type } G\,\vec{\alpha} = \Sigma(C : \forall \vec{\alpha}.\,[\beta \mapsto G\,\vec{\alpha}]\tau \to G\,\vec{\alpha}) \qquad \text{where } \vec{\alpha} = \text{ftv}(\tau) \smallsetminus \{\beta\}$$

For any $\vec{\alpha}$, the constructor $C\vec{\alpha}$ coerces $[\beta \mapsto G\,\vec{\alpha}]\tau$ to $G\,\vec{\alpha}$ and the reverse coercion is the function $\lambda x : G\,\vec{\alpha}.\,d_G\,\vec{\alpha}\,x\,(\lambda y.\,y)$.

Since this datatype has a unique constructor, pattern matching always succeeds and amounts to the identity. Hence, in $\lceil F \rceil$, the constructor could be removed: coercions have no computational content.

Records

A record can be defined as

$$\text{type } G\, \vec{\alpha} = \Pi_{i \in 1..n}(\ell_i : \tau_i) \qquad \text{where } \vec{\alpha} = \bigcup_{i \in 1..n} \text{ftv}(\tau_i)$$

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Can a record also be used for defining recursive types?

Exercise

Show type soundness for records.

Deep pattern matching

In practice, one allows deep pattern matching and wildcards in patterns.

```
type nat = Z | S of nat
let rec equal n1 n2 = match n1, n2 with
  | Z, Z → true
  | S m1, S m2 → equal m1 m2
  | _ → false
```

Then, one should check for *exhaustiveness* of pattern matching.

Deep pattern matching can be compiled away into shallow patterns—or directly compiled to efficient code.

See [Le Fessant and Maranget, 2001; Maranget, 2007]

Exercise
Do the transofrmation manually for the function equal.

ADTs   Regular

$$\text{type } G \ \vec{\alpha} = \Sigma_{i \in 1..n}(C_i : \forall \vec{\alpha}. \tau_i \to G \ \vec{\alpha})$$

If all occurrences of G in τ_i are $G \ \vec{\alpha}$ then, the ADT is *regular*.

Remark regular ADTs can be encoded in System-F. (More precisely, the church encodings of regular ADTs are typable in System-F.)

ADTs   Non Regular

Non-regular ADT's do not have this restriction:

```
type 'a seq =
  | Nil
  | Zero of ('a * 'a) seq
  | One of 'a * ('a * 'a) seq
```

They usually need *polymorphic* recursion to be manipulated.

Non regular ADT are heavily used by Okasaki [1999] for implementing purely functional data structures.

(They are also typically used with GADTs.)

Non-regular ADT can actually be encoded in F^ω.

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Existential types   *Examples*

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Type of closures in the environment-passing variant:

$$[\![\tau_1 \to \tau_2]\!] \quad = \quad \exists\alpha.((\alpha \times [\![\tau_1]\!]) \to [\![\tau_2]\!]) \times \alpha$$

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A possible encoding of objects:

$$
\begin{aligned}
= \quad \exists\rho. \qquad\qquad\qquad &\rho \text{ describes the state} \\
\mu\alpha. \qquad\qquad\qquad &\alpha \text{ is the concrete type of the closure} \\
\Pi\,(\qquad\qquad\qquad &\text{a tuple...} \\
\{(\alpha \times \tau_1) \to \tau_1'; \quad &\text{... that begins with a record...} \\
\cdots \qquad\qquad &\\
(\alpha \times \tau_n) \to \tau_n'\,\}\,; \quad &\text{... of method code pointers...} \\
\rho \qquad\qquad\quad &\text{...and continues with the state} \\
) \qquad\qquad\qquad &\quad \text{(a tuple of unknown length)}
\end{aligned}
$$

Existential types

Let's first look at the type-erasing interpretation, with an explicit notation for introducing and eliminating existential types.

Existential types in explicit style

Here is how the existential quantifier is introduced and eliminated:

PACK
$$\frac{\Gamma \vdash M : [\alpha \mapsto \tau']\tau}{\Gamma \vdash \text{pack}\, \tau', M \text{ as } \exists \alpha.\, \tau : \exists \alpha.\, \tau}$$

UNPACK
$$\frac{\Gamma \vdash M_1 : \exists \alpha.\tau_1 \qquad \Gamma, \alpha, x : \tau_1 \vdash M_2 : \tau_2}{\Gamma \vdash \text{let } \alpha, x = \text{unpack } M_1 \text{ in } M_2 : \tau_2}$$

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$$\frac{\Gamma \vdash M : [\alpha \mapsto \tau']\tau}{\Gamma \vdash \text{\textit{pack}}\ \tau', M \text{ \textit{as}}\ \exists \alpha.\, \tau : \exists \alpha.\, \tau}$$

UNPACK
$$\frac{\Gamma \vdash M_1 : \exists \alpha.\tau_1 \qquad \Gamma, \alpha, x : \tau_1 \vdash M_2 : \tau_2}{\Gamma \vdash \text{\textit{let}}\ \alpha, x = \text{\textit{unpack}}\ M_1 \text{ \textit{in}}\ M_2 : \tau_2}$$

Anything wrong?

Existential types in explicit style

Here is how the existential quantifier is introduced and eliminated:

PACK
$$\frac{\Gamma \vdash M : [\alpha \mapsto \tau']\tau}{\Gamma \vdash \textit{pack}\ \tau', M\ \textit{as}\ \exists \alpha.\tau : \exists \alpha.\tau}$$

UNPACK
$$\frac{\Gamma \vdash M_1 : \exists \alpha.\tau_1 \qquad \Gamma, \alpha, x : \tau_1 \vdash M_2 : \tau_2 \qquad \alpha \# \tau_2}{\Gamma \vdash \textit{let}\ \alpha, x = \textit{unpack}\ M_1\ \textit{in}\ M_2 : \tau_2}$$

The side condition $\alpha \# \tau_2$ is mandatory here to ensure well-formedness of the conclusion.

The side condition may also be written $\Gamma \vdash \tau_2$ which implies $\alpha \# \tau_2$, given that the well-formedness of the last premise implies $\alpha \notin \mathrm{dom}(\Gamma)$.

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$$
\begin{array}{c}
\text{Pack} \\
\dfrac{\Gamma \vdash M : [\alpha \mapsto \tau']\tau}{\Gamma \vdash pack\ \tau', M\ as\ \exists\alpha.\tau : \exists\alpha.\tau}
\end{array}
\qquad
\begin{array}{c}
\text{Unpack} \\
\Gamma \vdash M_1 : \exists\alpha.\tau_1 \\
\dfrac{\Gamma, \alpha, x : \tau_1 \vdash M_2 : \tau_2 \qquad \alpha\ \#\ \tau_2}{\Gamma \vdash let\ \alpha, x = unpack\ M_1\ in\ M_2 : \tau_2}
\end{array}
$$

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Note the *imperfect* duality between universals and existentials:

$$
\begin{array}{c}
\text{TAbs} \\
\dfrac{\Gamma, \alpha \vdash M : \tau}{\Gamma \vdash \Lambda\alpha.\,M : \forall\alpha.\tau}
\end{array}
\qquad\qquad
\begin{array}{c}
\text{TApp} \\
\dfrac{\Gamma \vdash M : \forall\alpha.\tau}{\Gamma \vdash M\ \tau' : [\alpha \mapsto \tau']\tau}
\end{array}
$$

On existential elimination

It would be nice to have a simpler elimination form, perhaps like this:

$$\frac{\Gamma, \alpha \vdash M : \exists \alpha.\tau}{\Gamma, \alpha \vdash \textit{unpack } M : \tau}$$

Informally, this could mean that, if M has type τ for some *unknown* α, then it has type τ, where α is "fresh"...

Why is this broken?

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Why is this broken?

We could immediately *universally* quantify over α, and conclude that $\Gamma \vdash \Lambda \alpha.\,\text{\textit{unpack }} M : \forall \alpha.\,\tau$. This is nonsense!

Replacing the premise $\Gamma, \alpha \vdash M : \exists \alpha.\tau$ by the conjunction $\Gamma \vdash M : \exists \alpha.\tau$ and $\alpha \in \mathrm{dom}(\Gamma)$ would make the rule even more permissive, so it wouldn't help.

On existential elimination

A correct elimination rule must force the existential package to be *used* in a way that does not rely on the value of α.

Hence, the elimination rule must have control over the *user* of the package—that is, over the term M_2.

$$
\begin{array}{c}
\text{Unpack} \\
\Gamma \vdash M_1 : \exists \alpha . \tau_1 \\
\dfrac{\Gamma, \alpha; x : \tau_1 \vdash M_2 : \tau_2 \qquad \alpha \mathbin{\#} \tau_2}{\Gamma \vdash \text{let } \alpha, x = \text{unpack } M_1 \text{ in } M_2 : \tau_2}
\end{array}
$$

The restriction $\alpha \mathbin{\#} \tau_2$ prevents writing "*let* $\alpha, x = $ *unpack* M_1 *in* x", which would be equivalent to the unsound "*unpack* M" of the previous slide.

The fact that α is bound within M_2 forces it to be treated abstractly.

In fact, M_2 must be   ??? in α.

On existential elimination

In fact, M_2 must be *polymorphic* in α: the second premise could be:

$$\frac{\Gamma \vdash M_1 : \exists \alpha.\tau_1 \qquad \Gamma,\ \alpha, x : \tau_1\ \vdash M_2 : \qquad \tau_2 \qquad \alpha \mathbin{\#} \tau_2}{\Gamma \vdash \textit{let } \alpha, x = \textit{unpack } M_1 \textit{ in } M_2 : \tau_2}$$

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or, if N_2 stands for $\Lambda\alpha.\,\lambda x{:}\tau_1.\, M_2$:

$$\frac{\Gamma \vdash M_1 : \exists \alpha.\tau_1 \qquad \Gamma \vdash N_2 : \forall \alpha.\,\tau_1 \to \tau_2 \qquad \alpha \,\#\, \tau_2}{\Gamma \vdash \textit{unpack } M_1 \; N_2 : \tau_2}$$

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One could even view "$\textit{unpack}_{\exists\alpha.\tau_1}$" as a family of *constants* of types:

$$\textit{unpack}_{\exists\alpha.\tau_1} : \quad (\exists\alpha.\tau_1) \to \big(\forall\alpha.\,(\tau_1 \to \tau_2)\big) \to \tau_2 \qquad \alpha \mathbin{\#} \tau_2$$

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Thus, $\qquad \textit{unpack}_{\exists\alpha.\tau} : \quad \forall\beta.\big((\exists\alpha.\tau) \to (\forall\alpha.\,(\tau \to \beta)) \to \beta\big)$

β stands for τ_2: it is bound prior to α, so it cannot be instantiated to a type that refers to α, which reflects the side condition $\alpha \mathbin{\#} \tau_2$.

On existential elimination

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On existential introduction

$$\text{PACK} \atop \dfrac{\Gamma \vdash M : [\alpha \mapsto \tau']\tau}{\Gamma \vdash \text{pack }\tau', M \text{ as } \exists\alpha.\,\tau : \exists\alpha.\tau}$$

Hence, "$pack_{\exists\alpha.\tau}$" can be viewed as a family *constant* of types:

$$pack_{\exists\alpha.\tau} : \quad [\alpha \mapsto \tau']\tau \to \exists\alpha.\tau$$

i.e. of polymorphic types:

$$pack_{\exists\alpha.\tau} : \quad \forall\alpha.\,(\tau \to \exists\alpha.\tau)$$

Existentials as constants

In System F, existential types can be presented as a family of constants:

$$pack_{\exists\alpha.\tau} \quad : \quad \forall\alpha.\,(\tau \to \exists\alpha.\tau)$$
$$unpack_{\exists\alpha.\tau} \quad : \quad \exists\alpha.\,\tau \to \forall\beta.\,((\forall\alpha.\,(\tau \to \beta)) \to \beta)$$

Read:

- for *any* α, if you have a τ, then, for *some* α, you have a τ;
- if, for *some* α, you have a τ, then, (for any β,) if you wish to obtain a β out of it, you must present a function which, for *any* α, obtains a β out of a τ.

This is somewhat reminiscent of ordinary first-order logic:
$\exists x.F$ is equivalent to, and can be defined as, $\neg(\forall x.\,\neg F)$.

Is there an encoding of existential types into universal types?

Encoding existentials into universals

The type translation is *double negation:*

$$[\![\exists\alpha.\tau]\!] \;=\; \forall\beta.\,((\forall\alpha.\,([\![\tau]\!]\to\beta))\to\beta) \qquad \text{if } \beta \,\#\, \tau$$

The term translation is:

$$
\begin{aligned}
[\![pack_{\exists\alpha.\tau}]\!] &\;:\; \forall\alpha.\,([\![\tau]\!]\to[\![\exists\alpha.\tau]\!]) \\
&\;=\; ? \\
[\![unpack_{\exists\alpha.\tau}]\!] &\;:\; [\![\exists\alpha.\tau]\!]\to\forall\beta.\,((\forall\alpha.\,([\![\tau]\!]\to\beta))\to\beta) \\
&\;=\; ?
\end{aligned}
$$

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$$
\begin{aligned}
[\![pack_{\exists\alpha.\tau}]\!] &: \forall\alpha.\left([\![\tau]\!] \to [\![\exists\alpha.\tau]\!]\right) \\
&= \Lambda\alpha.\,\lambda x\!:\![\![\tau]\!].\; ? \\
[\![unpack_{\exists\alpha.\tau}]\!] &: [\![\exists\alpha.\tau]\!] \to \forall\beta.\left((\forall\alpha.([\![\tau]\!] \to \beta)) \to \beta\right) \\
&= \;?
\end{aligned}
$$

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The term translation is:

$$
\begin{aligned}
\llbracket pack_{\exists \alpha.\tau} \rrbracket \;:\;& \forall \alpha.\,(\llbracket \tau \rrbracket \to \llbracket \exists \alpha.\tau \rrbracket) \\
=\;& \Lambda \alpha.\,\lambda x\!:\!\llbracket \tau \rrbracket.\,\Lambda \beta.\,\lambda k\!:\!\forall \alpha.\,(\llbracket \tau \rrbracket \to \beta).\,?:\beta \\
\llbracket unpack_{\exists \alpha.\tau} \rrbracket \;:\;& \llbracket \exists \alpha.\tau \rrbracket \to \forall \beta.\,((\forall \alpha.\,(\llbracket \tau \rrbracket \to \beta)) \to \beta) \\
=\;& ?
\end{aligned}
$$

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$$
\begin{aligned}
[\![pack_{\exists\alpha.\tau}]\!] \;&:\; \forall\alpha.\,([\![\tau]\!]\to[\![\exists\alpha.\tau]\!]) \\
&=\; \Lambda\alpha.\,\lambda x{:}[\![\tau]\!].\,\Lambda\beta.\,\lambda k{:}\forall\alpha.\,([\![\tau]\!]\to\beta).\,k\;\alpha\;x \\
[\![unpack_{\exists\alpha.\tau}]\!] \;&:\; [\![\exists\alpha.\tau]\!]\to\forall\beta.\,((\forall\alpha.\,([\![\tau]\!]\to\beta))\to\beta) \\
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$$
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[\![unpack_{\exists\alpha.\tau}]\!] \quad &: \quad [\![\exists\alpha.\tau]\!] \to \forall\beta.\,((\forall\alpha.\,([\![\tau]\!] \to \beta)) \to \beta) \\
&= \quad \lambda x\mathbin{:}[\![\exists\alpha.\tau]\!].\,x
\end{aligned}
$$

Encoding existentials into universals

The type translation is *double negation:*

$$\llbracket \exists \alpha.\tau \rrbracket \;=\; \forall \beta.\left(\left(\forall \alpha.\left(\llbracket \tau \rrbracket \rightarrow \beta\right)\right) \rightarrow \beta\right) \qquad \text{if } \beta \mathbin{\#} \tau$$

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$$
\begin{aligned}
\llbracket pack_{\exists \alpha.\tau} \rrbracket \;&:\; \forall \alpha.\left(\llbracket \tau \rrbracket \rightarrow \llbracket \exists \alpha.\tau \rrbracket\right) \\
&=\; \Lambda \alpha.\,\lambda x\!:\!\llbracket \tau \rrbracket.\,\Lambda \beta.\,\lambda k\!:\!\forall \alpha.\left(\llbracket \tau \rrbracket \rightarrow \beta\right).\,k\;\alpha\;x \\[4pt]
\llbracket unpack_{\exists \alpha.\tau} \rrbracket \;&:\; \llbracket \exists \alpha.\tau \rrbracket \rightarrow \forall \beta.\left(\left(\forall \alpha.\left(\llbracket \tau \rrbracket \rightarrow \beta\right)\right) \rightarrow \beta\right) \\
&=\; \lambda x\!:\!\llbracket \exists \alpha.\tau \rrbracket.\,x
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$$

There is little choice, if the translation is to be type-preserving.

What is the computational content of this encoding?

Encoding existentials into universals

The type translation is *double negation:*

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\end{aligned}
$$

There is little choice, if the translation is to be type-preserving.

What is the computational content of this encoding?

A *continuation-passing transform.*

This encoding is due to Reynolds [1983],
although it has more ancient roots in logic.

The semantics of existential types   as constants

$pack_{\exists\alpha.\tau}$ can be treated as a unary constructor, and $unpack_{\exists\alpha.\tau}$ as a unary destructor. The δ-reduction rule is:

$$unpack_{\exists\alpha.\tau_0} \; (pack_{\exists\alpha.\tau}\tau'\,V) \;\longrightarrow\; \Lambda\beta.\,\lambda y{:}\forall\alpha.\,\tau\to\beta.\,y\,\tau'\,V$$

It would be more intuitive, however, to treat $unpack_{\exists\alpha.\tau_0}$ as a binary destructor:

$$unpack_{\exists\alpha.\tau_0} \; (pack_{\exists\alpha.\tau}\tau'\,V)\; \tau_1 \;(\Lambda\alpha.\,\lambda x{:}\tau.\,M) \;\longrightarrow\; [\alpha\mapsto\tau'][x\mapsto V]M$$

Remark:

- This does not quite fit in our generic framework for constants, which must receive all type arguments prior to value arguments.
- But our framework could be easily extended.

The semantics of existential types   as primitive

We extend values and evaluation contexts as follows:

$$V \quad ::= \quad \dots \quad | \quad pack\ \tau', V\ as\ \tau$$
$$E \quad ::= \quad \dots \quad | \quad pack\ \tau', [\ ]\ as\ \tau \quad | \quad let\ \alpha, x = unpack\ [\ ]\ in\ M$$

We add the reduction rule:

$$let\ \alpha, x = unpack\ (pack\ \tau', V\ as\ \tau)\ in\ M \longrightarrow [\alpha \mapsto \tau'][x \mapsto V]M$$

Exercise
Show that subject reduction and progress hold.

The semantics of existential types   beware!

The reduction rule for existentials destructs its arguments.

Hence, *let* α, x = *unpack* M_1 *in* M_2 cannot be reduced unless M_1 is itself a packed expression, which is indeed the case when M_1 is a value (or in head normal form).

This contrasts with *let* $x : \tau = M_1$ *in* M_2 where M_1 need not be evaluated and may be an application (*e.g.* with call-by-name or strong reduction strategies).

The semantics of existential types   beware!

Exercise
Find an example that illustrates why the reduction of
let α, x = unpack M_1 in M_2 could be problematic when M_1 is not a value.

The semantics of existential types   beware!

Exercise
Find an example that illustrates why the reduction of
let α, x = unpack M_1 in M_2 could be problematic when M_1 is not a value.

Need a hint?

Use a conditional

The semantics of existential types   beware!

Exercise
Find an example that illustrates why the reduction of
let α, x = unpack M_1 in M_2 could be problematic when M_1 is not a value.

Solution

Let M_1 be if M then V_1 else V_2 where V_i is of the form
pack τ_i, W_i as $\exists \alpha.\tau$ and the two witnesses τ_1 and τ_2 differ.

There is no common type for the unpacking of the two possible results
V_1 and V_2. The choice between those two possible results must be made,
by evaluating M_1, before unpacking.

Is pack too verbose?

Exercise
Recall the typing rule for pack:

$$\frac{\Gamma \vdash M : [\alpha \mapsto \tau']\tau}{\Gamma \vdash \textit{pack}\,\tau', M \textit{ as } \exists\alpha.\,\tau : \exists\alpha.\tau}$$

Isn't the witness type τ' annotation superfluous?

Is pack too verbose?

Exercise
Recall the typing rule for pack:

$$\frac{\Gamma \vdash M : [\alpha \mapsto \tau']\tau}{\Gamma \vdash \text{pack } \tau', M \text{ as } \exists\alpha.\tau : \exists\alpha.\tau}$$

Isn't the witness type τ' annotation superfluous?

- The type τ_0 of M is fully determined by M. Given the type $\exists\alpha.\tau$ of the packed value, checking that τ_0 is of the form $[\alpha \mapsto \tau']\tau$ is the matching problem for second-order types, which is simple.

- However, the reduction rule need the witness type τ'. If it were not available, it would have to be computed during reduction. The reduction rule would then not be pure rewriting.

The explicitly-typed language need the witness type for simplicity, while in the surface language, it could be omitted and reconstructed.

- Algebraic Data Types
  - Equi- and iso- recursive types

- Existential types
  - Implicitly-type existential types passing
  - Iso-existential types

- Generalized Algebraic Datatypes

- Application to typed closure conversion
  - Environment passing
  - Closure passing

Implicitly-typed existential types

Intuitively, pack and unpack are just type annotations that could be dropped, leaving a let-binding instead of the unpack form.

Hence, the typing rule for implicitly-typed existential types:

UNPACK
$$\frac{\Gamma \vdash a_1 : \exists \alpha.\tau_1 \qquad \Gamma, \alpha, x : \tau_1 \vdash a_2 : \tau_2 \qquad \alpha \mathbin{\#} \tau_2}{\Gamma \vdash let\ x = a_1\ in\ a_2 : \tau_2}$$

PACK
$$\frac{\Gamma \vdash a : [\alpha \mapsto \tau']\tau}{\Gamma \vdash a : \exists \alpha.\tau}$$

Notice, however, that this let-binding is not typechecked as syntactic sugar for an immediate application!

The semantics of this let-binding is as before:

$$E ::= \dots \mid let\ x = E\ in\ M \qquad let\ x = V\ in\ M \longrightarrow [x \mapsto V]M$$

Is the semantics type-erasing?

Implicitly-typed existential types   subtlety

Yes, it is.

But there is a subtlety!

Implicitly-typed existential types   subtlety

Yes, it is.

But there is a subtlety! What about the call-by-name semantics?

Implicitly-typed existential types   subtlety

Yes, it is.

But there is a subtlety! What about the call-by-name semantics?

We chose a call-by-value semantics, but so far, as long as there is no side-effect, we could have chosen a call-by-name semantics (or even perform reduction under abstraction).

In a call-by-name semantics, the let-bound expression is not reduced prior to substitution in the body:

$$\textit{let } x = M_1 \textit{ in } M_2 \longrightarrow [x \mapsto M_1]M_2$$

With existential types, this breaks subject reduction!

Why?

Implicitly-typed existential types   subtlety

Let τ_0 be $\exists \alpha. (\alpha \to \alpha) \to (\alpha \to \alpha)$ and v_0 a value of type *bool*. Let v_1 and v_2 be two values of type τ_0 with incompatible witness types, *e.g.* $\lambda f. \lambda x. 1 + (f (1 + x))$ and $\lambda f. \lambda x. not (f (not x))$.

Let v be the function $\lambda b.$ if b then v_1 else v_2 of type *bool* $\to \tau_0$.

$$a_1 \;=\; \textit{let } x = v \, v_0 \textit{ in } x \, (x \, (\lambda y. \, y)) \;\longrightarrow\; v \, v_0 \, (v \, v_0 \, (\lambda y. \, y)) \;=\; a_2$$

We have $\varnothing \vdash a_1 : \exists \alpha. \alpha \to \alpha$ while $\varnothing \nvdash a_2 : \tau$.

What happened?

Implicitly-typed existential types   subtlety

Let τ_0 be $\exists \alpha. (\alpha \to \alpha) \to (\alpha \to \alpha)$ and v_0 a value of type *bool*. Let v_1 and v_2 be two values of type τ_0 with incompatible witness types, *e.g.* $\lambda f. \lambda x. 1 + (f (1 + x))$ and $\lambda f. \lambda x. not (f (not x))$.

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$$a_1 \;=\; \textit{let } x = v \, v_0 \textit{ in } x \, (x \, (\lambda y. \, y)) \;\; \longrightarrow \;\; v \, v_0 \, (v \, v_0 \, (\lambda y. \, y)) \;=\; a_2$$

We have $\varnothing \vdash a_1 : \exists \alpha. \, \alpha \to \alpha$ while $\varnothing \nvdash a_2 : \tau$.

The term a_1 is well-typed since $v \, v_0$ has type τ_0, hence x can be assumed of type $(\beta \to \beta) \to (\beta \to \beta)$ for some unknown type β and $\lambda y. \, y$ is of type $\beta \to \beta$.

However, without the outer existential type $v \, v_0$ can only be typed with $(\forall \alpha. \, \alpha \to \alpha) \to \exists \alpha. \, (\alpha \to \alpha)$, because the value returned by the function need different witnesses for α. This is demanding too much on its argument and the outer application is ill-typed.

Implicitly-typed existential types   subtlety

One could wonder whether the syntax should not allow the implicit introduction of unpacking (instead of requesting a let-binding).

One could argue that if some expression is the expansion of a well-typed let-binding, then it should also be well-typed:

$$\frac{\Gamma \vdash a_1 : \exists \alpha. \tau_1 \qquad \Gamma, \alpha, x : \tau_1 \vdash a_2 : \tau_2 \qquad \alpha \;\#\; \tau_2}{\Gamma \vdash [x \mapsto a_1] a_2 : \tau_2}$$

Comments?

Implicitly-typed existential types   subtlety

One could wonder whether the syntax should not allow the implicit
introduction of unpacking (instead of requesting a let-binding).

One could argue that if some expression is the expansion of a well-typed
let-binding, then it should also be well-typed:

$$\frac{\Gamma \vdash a_1 : \exists \alpha.\tau_1 \qquad \Gamma, \alpha, x : \tau_1 \vdash a_2 : \tau_2 \qquad \alpha \mathrel{\#} \tau_2}{\Gamma \vdash [x \mapsto a_1]a_2 : \tau_2}$$

Comments:

- This rule does not have a logical flavor...
- It fixes the previous example, but not the general case:
  Pick a_1 that is not yet a value after one reduction step.
  Then, after let-expansion, reduce one of the two occurrences of a_1.
  The result is no longer of the form $[x \mapsto a_1]a_2$.

Implicitly-typed existential types   subtlety

Existential types are trickier than they may appear at first.

The subject reduction property breaks if reduction is not restricted to expressions in head-normal forms.

Unrestricted reduction is still safe because well-typedness may eventually be recovered by further reduction steps—so that progress will never breaks.

Implicitly-typed existential types   encoding

Notice that the CPS encoding of existential types (1) enforces the evaluation of the packed value (2) before it can be unpacked (3) and substituted (4):

$$
\begin{aligned}
\llbracket \mathit{unpack}\, a_1\, (\lambda x.\, a_2) \rrbracket &= \llbracket a_1 \rrbracket \, (\lambda x.\, \llbracket a_2 \rrbracket) & (\mathbf{1}) \\
&\longrightarrow (\lambda k.\, \llbracket a \rrbracket\, k)\, (\lambda x.\, \llbracket a_2 \rrbracket) & (\mathbf{2}) \\
&\longrightarrow (\lambda x.\, \llbracket a_2 \rrbracket)\, \llbracket a \rrbracket & (\mathbf{3}) \\
&\longrightarrow [x \mapsto \llbracket a \rrbracket] \llbracket a_2 \rrbracket & (\mathbf{4})
\end{aligned}
$$

In the call-by-value setting, $\lambda k.\, \llbracket a \rrbracket\, k$ would come from the reduction of $\llbracket \mathit{pack}\, a \rrbracket$, *i.e.* is $(\lambda k.\, \lambda x.\, k\, x)\, \llbracket a \rrbracket$, so that a is always a value v.

However, a need not be a value. What is essential is that a_1 be reduced to some head normal form $\lambda k.\, \llbracket a \rrbracket\, k$.

- Algebraic Data Types
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Iso-existential types in ML

What if one wished to extend ML with existential types?

Full type inference for existential types is undecidable, just like type inference for universals.

However, introducing existential types in ML is easy if one is willing to rely on user-supplied *annotations* that indicate *where* and *how* to pack and unpack.

Iso-existential types in ML

This *iso-existential* approach was suggested by Läufer and Odersky [1994].

Iso-existential types are explicitly *declared:*

$$D \, \vec{\alpha} \approx \exists \bar{\beta}.\tau \qquad \text{if } \mathrm{ftv}(\tau) \subseteq \bar{\alpha} \cup \bar{\beta} \quad \text{and} \quad \bar{\alpha} \, \# \, \bar{\beta}$$

This introduces two constants, with the following type schemes:

$$\begin{aligned}
\mathit{pack}_D &: \quad \forall \bar{\alpha}\bar{\beta}. \, \tau \to D \, \vec{\alpha} \\
\mathit{unpack}_D &: \quad \forall \bar{\alpha}\gamma. \, D \, \vec{\alpha} \to (\forall \bar{\beta}. \, (\tau \to \gamma)) \to \gamma
\end{aligned}$$

(Compare with basic isorecursive types, where $\bar{\beta} = \varnothing$.)

Iso-existential types in ML

One point has been hidden on the previous slide. The "type scheme:"

$$\forall \bar{\alpha}\gamma.\, D\ \vec{\alpha} \to (\forall \bar{\beta}.\,(\tau \to \gamma)) \to \gamma$$

is in fact *not* an ML type scheme. How could we address this?

Iso-existential types in ML

One point has been hidden on the previous slide. The "type scheme:"

$$\forall \bar{\alpha}\gamma.\, D\,\vec{\alpha} \to (\forall \bar{\beta}.\,(\tau \to \gamma)) \to \gamma$$

is in fact *not* an ML type scheme. How could we address this?

A solution is to make $unpack_D$ a (binary) primitive construct again (rather than a constant), with an *ad hoc* typing rule:

Unpack_D

$$\frac{\Gamma \vdash M_1 : D\,\vec{\tau} \qquad\qquad \Gamma \vdash M_2 : \forall \bar{\beta}.\,([\vec{\alpha} \mapsto \vec{\tau}]\tau \to \tau_2) \qquad \bar{\beta}\,\#\,\vec{\tau}, \tau_2}{\Gamma \vdash unpack_D\ M_1\ M_2 : \tau_2} \qquad \text{where } D\,\vec{\alpha} \approx \exists\bar{\beta}.\tau$$

We have seen a version of this rule in System F earlier; this is an ML(-like) version.

The term M_2 must be polymorphic, which GEN can prove.

Iso-existential types in ML (type inference, skip)

Iso-existential types are perfectly compatible with ML type inference.

The constant $pack_D$ admits an ML type scheme, so it is unproblematic.

The construct $unpack_D$ leads to this constraint generation rule (see type inference):

$$\langle\!\langle unpack_D\ M_1\ M_2 : \tau_2 \rangle\!\rangle \;\; = \;\; \exists\bar{\alpha}. \left(\begin{array}{l} \langle\!\langle M_1 : D\ \bar{\alpha} \rangle\!\rangle \\ \forall\bar{\beta}.\, \langle\!\langle M_2 : \tau \to \tau_2 \rangle\!\rangle \end{array} \right)$$

where $D\ \bar{\alpha} \approx \exists\bar{\beta}.\tau$ and, w.l.o.g., $\bar{\alpha}\bar{\beta} \mathrel{\#} M_1, M_2, \tau_2$.

A universally quantified constraint appears where polymorphism is *required*.

Iso-existential types in ML

In practice, Läufer and Odersky suggest fusing iso-existential types with algebraic data types.

This can be done in OCaml using GADTs (see last part of the course). The syntax for this in OCaml is:

$$type\ D\ \vec{\alpha} = \ell : \tau \to D\ \vec{\alpha}$$

where ℓ is a data constructor and $\bar{\beta}$ appears free in τ but does not appear in $\vec{\alpha}$. The elimination construct is typed as:

$$\langle\!\langle match\ M_1\ with\ \ell\ x \to M_2 : \tau_2 \rangle\!\rangle \;\; = \;\; \exists\bar{\alpha}.\left(\begin{array}{l} \langle\!\langle M_1 : D\ \vec{\alpha} \rangle\!\rangle \\ \forall\bar{\beta}.\ def\ x : \tau\ in\ \langle\!\langle M_2 : \tau_2 \rangle\!\rangle \end{array} \right)$$

where, w.l.o.g., $\bar{\alpha}\bar{\beta} \mathrel{\#} M_1, M_2, \tau_2$.

Existential types calls for universal types!

Exercise Let *thunk* $\alpha \approx \exists\beta.(\beta \to \alpha) \times \beta$ be the type of frozen computations. Assume given a list l with elements of type *thunk* τ_1.

Assume given a function g of type $\tau_1 \to \tau_2$. Transform the list l into a new list l' of frozen computations of type *thunk* τ_2 (without actually running any computation).

List.map $(\lambda(z)$ **let** Delay (f, y) = z **in** Delay $((\lambda(z)$ g (f z)), y))

Try generalizing this example to a function that receives g and l and returns l'

?

Existential types calls for universal types!

Exercise Let $thunk\ \alpha \approx \exists\beta.(\beta \to \alpha) \times \beta$ be the type of frozen computations. Assume given a list l with elements of type $thunk\ \tau_1$.

Assume given a function g of type $\tau_1 \to \tau_2$. Transform the list l into a new list l' of frozen computations of type $thunk\ \tau_2$ (without actually running any computation).

　　List.map $(\lambda(z)$ **let** Delay (f, y) = z **in** Delay $((\lambda(z)\ g\ (f\ z)),\ y))$

Try generalizing this example to a function that receives g and l and returns l' : it does not typecheck. . .

　　let lift g l =
　　　　List.map $(\lambda(z)$ **let** Delay (f, y) = z **in** Delay $((\lambda(z)\ g\ (f\ z)),\ y))$

<div align="center">?</div>

Existential types calls for universal types!

Exercise Let $thunk\,\alpha \approx \exists\beta.(\beta \to \alpha) \times \beta$ be the type of frozen computations. Assume given a list l with elements of type $thunk\,\tau_1$.

Assume given a function g of type $\tau_1 \to \tau_2$. Transform the list l into a new list l' of frozen computations of type $thunk\,\tau_2$ (without actually running any computation).

   List.map $(\lambda(z)$ **let** Delay (f, y) = z **in** Delay $((\lambda(z)\ g\ (f\ z)),\ y))$

Try generalizing this example to a function that receives g and l and returns l' : it does not typecheck. . .

   let lift g l =
      List.map $(\lambda(z)$ **let** Delay (f, y) = z **in** Delay $((\lambda(z)\ g\ (f\ z)),\ y))$

In expression *let* $\alpha, x = unpack\ M_1\ in\ M_2$, occurrences of x in M_2 can only be passed to external functions (free variables) that are polymorphic in α so that α does not leak out of its context.

Limits of iso-encodings

Using datatypes for existential and especially universal types is a simple solution to make them compatible with ML, but it comes with some limitations:

- All types must be declared before being used
- Programs become quite verbose, with many constructors that amount to writting type annotations, but in a more rigid way
- In particular, there is no canonical way of representing them. For exemple, a thunk of type $\exists \beta (\beta \rightarrow int) \times \beta$ could have been defined as Delay (succ, 1) where Delay is either one of

  type int_thunk = Delay : ('b → int) * 'b → int_thunk
  type 'a thunk = Delay : ('b → 'a) * 'b → 'a thunk

  but the two types are incompatible.

Hence, other primitive solutions have been considered, especially for universal types.

Uses of existential types

Mitchell and Plotkin [1988] note that existential types offer a means of explaining *abstract types.* For instance, the type:

$$\exists stack.\{empty : stack;$$
$$push : int \times stack \rightarrow stack;$$
$$pop : stack \rightarrow option\,(int \times stack)\}$$

specifies an abstract implementation of integer stacks.

Unfortunately, it was soon noticed that the elimination rule is too awkward, and that existential types alone do not allow designing *module systems* [Harper and Pierce, 2005].

Montagu and Rémy [2009] make existential types *more flexible* in several important ways, and argue that they might explain modules after all.

Rossberg, Russo, and Dreyer show that after all, generative modules can be encoded into System F with existential types [Rossberg et al., 2014].

Existential types in OCaml

Existential types are available indirectly in OCaml as a degenerate case of GADT and via abstract types and first-class modules.

Via GADT (iso-existential types)

```
type 'a thunk = Delay : ('b → 'a) * 'b → 'a thunk
let freeze f x = Delay (f, x)
let unfreeze (Delay (f, x)) = f x
```

Via first-class modules (abstract types)

```
module type Thunk = sig type b type a val f : b → a val x : b end
let freeze (type u) (type v) f x =
   (module struct type b = u type a = v  let f = f let x = x end
    : Delay)
let unfreeze (type u) (module M : Thunk with type a = u) = M.f M.x
```

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An introduction to GADTs

What are they?

ADTs

Types of constructors are surjective: all types can potentially be reached

    type α list $=$
      | Nil : α list
      | Const : $\alpha * \alpha$ list $\rightarrow \alpha$ list

GADTs

This is no more the case with GADTs

    type (α, β) eq $=$
      | *Eq* : (α, α) eq

The *Eq* constructor may only build values of types of (α, α) eq.
For example, it cannot build values of type $(\text{int}, \text{string})$ eq.

What are they?

ADTs

Types of constructors are surjective: all types can potentially be reached

type α list =
   | Nil : α list
   | Const : $\alpha * \alpha$ list $\rightarrow \alpha$ list

GADTs

This is no more the case with GADTs

type (α, β) eq =
   | *Eq* : (α, α) eq
   | Any : (α, β) eq

The *Eq* constructor may only build values of types of (α, α) eq.
For example, it cannot build values of type $(\text{int}, \text{string})$ eq.

The criteria is *per constructor*: it remains a GADT when another (even
regular) constructor is added.

Examples   Defunctionalization

```
let add (x, y) = x + y in
let not x = if x then false else true in
let body b =
   let step x =
     add (x, if not b then 1 else 2)
   in step (step 0))
in body true
```

Examples   Defunctionalization

```
let add (x, y) = x + y in
let not x = if x then false else true in
let body b =
    let step x =

      add (x, if not b then 1 else 2)
    in step (step 0))
in body true
```

Introduce a constructor per function

```
type (_, _) apply =
| Fadd  : (int * int, int) apply
| Fnot  : (bool, bool) apply
| Fbody : (bool, int) apply
| Fstep : bool → (int, int) apply
```

Examples   Defunctionalization

```
let add (x, y) = x + y in
let not x = if x then false else true in
let body b =
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Introduce a constructor per function

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type (_, _) apply =
| Fadd  : (int * int, int) apply
| Fnot  : (bool, bool) apply
| Fbody : (bool, int) apply
| Fstep : bool → (int, int) apply
```

Define a single apply function that dispatches all function calls:

```
let rec apply : type a b. (a, b) apply → a → b =  fun f arg →
  match f with
  | Fadd    → let x, y = arg in x + y
  | Fnot    → let x = arg in if x then false else true
  | Fstep b → let x = arg in
              apply Fadd (x, if apply Fnot b then 1 else 2)
  | Fbody   → let b = arg in
              apply (Fstep b) (apply (Fstep b) 0)
in apply Fbody true
```

Examples   Defunctionalization

```
let add (x, y) = x + y in
let not x = if x then false else true in
let body b =
    let step x =
      add (x, if not b then 1 else 2)
    in step (step 0))
in body true
```

Introduce a constructor per function

```
type (_, _) apply =
| Fadd  : (int * int, int) apply
| Fnot  : (bool, bool) apply
| Fbody : (bool, int) apply
| Fstep : bool → (int, int) apply
```

Key point: the typechecker refines the types a and b in each branch

```
let rec apply : type a b. (a, b) apply → a → b =  fun f arg →
  match f with                                  (* a =         b = *)
  | Fadd    → let x, y = arg in x + y            (* int * int   int  *)
  | Fnot    → let x = arg in if x then false else true  (* bool        bool *)
  | Fstep b → let x = arg in                     (* int         int  *)
               apply Fadd (x, if apply Fnot b then 1 else 2)
  | Fbody   → let b = arg in                      (* bool        int  *)
               apply (Fstep b) (apply (Fstep b) 0)
  in apply Fbody true
```

Examples   Typed evaluator

A typed abstract-syntax tree

```
type _ expr =
| Int    : int → int expr
| Zerop  : int expr → bool expr
| If     : (bool expr * α expr * α expr) → α expr
let e0         = (If (Zerop (Int 0), Int 1, Int 2))
```

What is the type of e0?

Examples   Typed evaluator

A typed abstract-syntax tree

```
type _ expr =
  | Int    : int → int expr
  | Zerop  : int expr → bool expr
  | If     : (bool expr ∗ α expr ∗ α expr) → α expr
let e0 : int expr =  (If (Zerop (Int 0), Int 1, Int 2))
```

A typed evaluator (with no failure)

```
let rec eval : type a . a expr → a = fun x → match x with
  | Int x          → x                         (∗ a = int ∗)
  | Zerop x        → eval x > 0                 (∗ a = bool ∗)
  | If (b, e1, e2) → if eval b then eval e1 else eval e2
let b0 = eval e0
```

Examples   Typed evaluator

A typed abstract-syntax tree

```
type _ expr =
| Int    : int → int expr
| Zerop  : int expr → bool expr
| If     : (bool expr * α expr * α expr) → α expr
let e0 : int expr =  (If (Zerop (Int 0), Int 1, Int 2))
```

A typed evaluator (with no failure)

```
let rec eval : type a . a expr → a = fun x → match x with
| Int x          → x                              (* a = int *)
| Zerop x        → eval x > 0                      (* a = bool *)
| If (b, e1, e2) → if eval b then eval e1 else eval e2
let b0 = eval e0
```

Exercise
What would you have to do without GADTs? Define a typed abstract
syntax tree for the simply-typed λ-calculus and a *typed* evaluator.

Examples   Generic programming

Example of printing

```
type _ ty =
  | Tint : int ty
  | Tbool : bool ty
  | Tlist : α ty → (α list) ty
  | Tpair : α ty * β ty → (α * β) ty

let rec to_string : type a. a ty → a → string = fun t x → match t with
  | Tint → string_of_int x
  | Tbool → if x then "true" else "false"
  | Tlist t → "[" ^ String.concat "; " (List.map (to_string t) x) ^ "]"
  | Tpair (a, b) →
      let u, v = x in "(" ^ to_string a u ^ ", " ^ to_string b v ^ ")"

let s = to_string (Tpair (Tlist Tint, Tbool)) ([1; 2; 3], true)
```

Examples   Encoding sum types

type (α, β) sum = Left of α | Right of β

can be encoded as a product:

type $(\_, \_, \_)$ tag = Ltag : (α, α, β) tag | Rtag : (β, α, β) tag
type (α, β) prod = Prod : (γ, α, β) tag $* \gamma \to (\alpha, \beta)$ prod

let sum_of_prod (**type** a b) (p : (a, b) prod) : (a, b) sum =
  let Prod (t, v) = p **in** match t with Ltag $\to$ Left v | Rtag $\to$ Right v

Prod is a single, hence superfluous constructor: it need not be allocated.

A field common to both cases can be accessed without looking at the tag!

type (α, β) prod = Prod : (γ, α, β) tag $* \gamma *$ bool $\to (\alpha, \beta)$ prod
let get (**type** a b) (p : (a, b) prod) : bool =
  let Prod (t, v, s) = p **in** s

Examples   Encoding sum types

Exercise

Specialize the encoding of sum types to the encoding of 'a list

Other uses of GADTs

GADTs

- May encode data-structure invariants, such as the state of an automaton, as illustrated by Pottier and Régis-Gianas [2006] for typechecking LR-parsers.

- They may be used to implement a form of dynamic type (similarly to the generic printer)

- They may be used to optimize representation (*e.g.* sum's encoding)

- GADTs can be used to encode type classes, using a technique analogous to defunctionalization [Pottier and Gauthier, 2006].

Reducing GADTs to type equality   (and existential types)

All GADTs can be encoded with a single one, encoding type equality:

  type (α, β) eq $= Eq : (\alpha, \alpha)$ eq

For instance, generic programming can then be redefined as follows:

  type α ty $=$
  | Tint : $(\alpha,$ int$)$ eq $\rightarrow \alpha$ ty   (* int ty   *)
  | Tlist : $(\alpha, \beta$ list$)$ eq $* \beta$ ty $\rightarrow \alpha$ ty   (* α ty $\rightarrow \alpha$ list ty *)
  | Tpair : $(\alpha, (\beta * \gamma))$ eq $* \beta$ ty $* \gamma$ ty $\rightarrow \alpha$ ty

This declaration is not a GADT, just an existential type!

  ▷ We enlarge the domain of each constructor,

  ▷ But require a proof evidence as an extra argument that a certain
    equality holds *to restrict the possible uses* of the constructors.

Reducing GADTs to type equality   (and existential types)

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type (α, β) eq = Eq : (α, α) eq

For instance, generic programming can then be redefined as follows:

```
type α ty =
  | Tint  : (α, int) eq → α ty                    (* int ty           *)
  | Tlist : (α, β list) eq * β ty → α ty          (* α ty → α list ty *)
  | Tpair : (α, (β * γ)) eq * β ty * γ ty → α ty
```

This declaration is not a GADT, just an existential type!

```
let rec to_string : type a. a ty → a → string = fun t x → match t with
  | Tint Eq        → string_of_int x
  | Tlist (Eq, l)  → "[" ^String.concat "; " (List.map (to_string l) x)^ "]"
  | Tpair (Eq,a,b) →
      let u, v = x in "(" ^ to_string a u ^ ", " ^ to_string b v ^ ")"

let s = to_string (Tpair (Eq, Tlist (Eq, Tint Eq), Tint Eq)) ([1; 2; 3], 0)
```

Reducing GADTs to type equality   (and existential types)

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This declaration is not a GADT, just an existential type!

>   **let rec** to_string : **type** a. a ty $\rightarrow$ a $\rightarrow$ string $=$ **fun** t x $\rightarrow$ **match** t **with**
> | Tint $Eq \rightarrow$ string_of_int x
> | Tlist $(Eq,$ l$) \rightarrow$ ...
> | Tpair $(Eq,$ a, b$) \rightarrow$ ...
>
>   ▷ Pattern "Tint Eq" is   GADT matching

Reducing GADTs to type equality   (and existential types)

All GADTs can be encoded with a single one :

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For instance, generic programming can be redefined as follows:

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  | Tpair : $(\alpha, (\beta * \gamma))$ eq $* \beta$ ty $* \gamma$ ty $\rightarrow \alpha$ ty

This declaration is not a GADT, just an existential type!

let rec to_string : **type** a. a ty $\rightarrow$ a $\rightarrow$ string $=$ **fun** t x $\rightarrow$ match t with
  | Tint p $\rightarrow$ **let** $Eq =$ p **in** string_of_int x
  | Tlist $(Eq,$ l$) \rightarrow$ ...
  | Tpair $(Eq,$ a, b$) \rightarrow$ ...

▷ Pattern "Tint p" is ordinary ADT matching

▷ **let** $Eq =$ p **in**.. introduces the equality a $=$ int in the current branch

Formalisation of GADTs

We can *extend* System F with type equalities to encode GADTs.

We *cannot* encode type equalities in System F—but in System F^ω: they bring something more, namely *local equalities* in the typing context.

We write $\tau_1 \sim \tau_2$ for (τ_1, τ_2) eq

When typechecking an expression

$$E[\text{let } x : \tau_1 \sim \tau_2 = M_0 \text{ in } M] \qquad E[\lambda x : \tau_1 \sim \tau_2. M]$$

▷ M is typechecked with the asumption that $\tau_1 \sim \tau_2$, *i.e.* types τ_1 and τ_2 are equivalent, which allows for type conversion within M

▷ but E and M_0 are typechecked without this asumption

▷ What is learned by an equation remains local to its static scope, and does not extend to its surrounding context (or the rest of the program execution trace).

Fc (simplified)   *Add equality coercions to System F*

Coercions witness type equivalences:

Types

$$\tau ::= \dots \mid \tau_1 \sim \tau_2$$

Expressions

$$M ::= \dots \mid \gamma \lhd M \mid \gamma$$

Coercions are first-class and can be applied to terms.

Typing rules:

$$
\begin{array}{ll}
\gamma ::= \alpha & \text{variable} \\
\mid \langle \tau \rangle & \text{reflexivity} \\
\mid \text{sym}\, \gamma & \text{symmetry} \\
\mid \gamma_1 ; \gamma_2 & \text{transitivity} \\
\mid \gamma_1 \to \gamma_2 & \text{arrow coercions} \\
\mid \text{left}\, \gamma & \text{left projection} \\
\mid \text{right}\, \gamma & \text{right projection} \\
\mid \forall \alpha.\, \gamma & \text{type generalization} \\
\mid \gamma @ \tau & \text{type instantiation}
\end{array}
$$

COERCE
$$\frac{\Gamma \vdash M : \tau_1 \quad \Gamma \vdash \gamma : \tau_1 \sim \tau_2}{\Gamma \vdash \gamma \lhd M : \tau_2}$$

COERCION
$$\frac{\Gamma \Vdash \gamma : \tau_1 \sim \tau_2}{\Gamma \vdash \gamma : \tau_1 \sim \tau_2}$$

COABS
$$\frac{\Gamma, x : \tau_1 \sim \tau_2 \vdash M : \tau}{\Gamma \vdash \lambda x : \tau_1 \sim \tau_2.\, M : \tau_1 \sim \tau_2 \to \tau}$$

Fc (simplified)   Typing of coercions

$$
\text{E{\scriptsize Q}-H{\scriptsize YP}} \\
\frac{y : \tau_1 \sim \tau_2 \in \Gamma}{\Gamma \Vdash y : \tau_1 \sim \tau_2}
$$

$$
\text{E{\scriptsize Q}-R{\scriptsize EF}} \\
\frac{\Gamma \vdash \tau}{\Gamma \Vdash \langle \tau \rangle : \tau \sim \tau}
$$

$$
\text{E{\scriptsize Q}-S{\scriptsize YM}} \\
\frac{\Gamma \Vdash \gamma : \tau_1 \sim \tau_2}{\Gamma \Vdash \mathsf{sym}\, \gamma : \tau_2 \sim \tau_1}
$$

$$
\text{E{\scriptsize Q}-T{\scriptsize RANS}} \\
\frac{\Gamma \Vdash \gamma_1 : \tau_1 \sim \tau \qquad \Gamma \Vdash \gamma_2 : \tau \sim \tau_2}{\Gamma \Vdash \gamma_1 ; \gamma_2 : \tau_1 \sim \tau_2}
$$

$$
\text{E{\scriptsize Q}-A{\scriptsize RROW}} \\
\frac{\Gamma \Vdash \gamma_1 : \tau_1' \sim \tau_1 \qquad \Gamma \Vdash \gamma_2 : \tau_2 \sim \tau_2'}{\Gamma \Vdash \gamma_1 \rightarrow \gamma_2 : \tau_1 \rightarrow \tau_2 \sim \tau_1' \rightarrow \tau_2'}
$$

$$
\text{E{\scriptsize Q}-L{\scriptsize EFT}} \\
\frac{\Gamma \Vdash \gamma : \tau_1 \rightarrow \tau_2 \sim \tau_1' \rightarrow \tau_2'}{\Gamma \Vdash \mathsf{left}\, \gamma : \tau_1' \sim \tau_1}
$$

$$
\text{E{\scriptsize Q}-R{\scriptsize IGHT}} \\
\frac{\Gamma \Vdash \gamma : \tau_1 \rightarrow \tau_2 \sim \tau_1' \rightarrow \tau_2'}{\Gamma \Vdash \mathsf{right}\, \gamma : \tau_2 \sim \tau_2'}
$$

$$
\text{E{\scriptsize Q}-A{\scriptsize LL}} \\
\frac{\Gamma, \alpha \Vdash \gamma : \tau_1 \sim \tau_2}{\Gamma \Vdash \forall \alpha.\, \gamma : \forall \alpha.\, \tau_1 \sim \forall \alpha.\, \tau_2}
$$

$$
\text{E{\scriptsize Q}-I{\scriptsize NST}} \\
\frac{\Gamma \Vdash \gamma : \forall \alpha.\, \tau_1 \sim \forall \alpha.\, \tau_2 \qquad \Gamma \vdash \tau}{\Gamma \Vdash \gamma@\tau : [\alpha \mapsto \tau]\tau_1 \sim [\alpha \mapsto \tau]\tau_2}
$$

Fc (simplified)   Typing of coercions

$$
\frac{\text{Eq-Hyp}}{\Gamma \Vdash y : \tau_1 \sim \tau_2 \in \Gamma}{\Gamma \Vdash y : \tau_1 \sim \tau_2}
\qquad
\frac{\text{Eq-Ref}}{\Gamma \vdash \tau}{\Gamma \Vdash \langle \tau \rangle : \tau \sim \tau}
\qquad
\frac{\text{Eq-Sym}}{\Gamma \Vdash \gamma : \tau_1 \sim \tau_2}{\Gamma \Vdash \mathsf{sym}\,\gamma : \tau_2 \sim \tau_1}
$$

$$
\frac{\text{Eq-Trans}}{\Gamma \Vdash \gamma_1 : \tau_1 \sim \tau \qquad \Gamma \Vdash \gamma_2 : \tau \sim \tau_2}{\Gamma \Vdash \gamma_1 ; \gamma_2 : \tau_1 \sim \tau_2}
\qquad
\frac{\text{Eq-Arrow}}{\Gamma \Vdash \gamma_1 : \tau_1' \sim \tau_1 \qquad \Gamma \Vdash \gamma_2 : \tau_2 \sim \tau_2'}{\Gamma \Vdash \gamma_1 \to \gamma_2 : \tau_1 \to \tau_2 \sim \tau_1' \to \tau_2'}
$$

$$
\frac{\text{Eq-Left}}{\Gamma \Vdash \gamma : \tau_1 \to \tau_2 \sim \tau_1' \to \tau_2'}{\Gamma \Vdash \mathsf{left}\,\gamma : \tau_1' \sim \tau_1}
\qquad
\frac{\text{Eq-Right}}{\Gamma \Vdash \gamma : \tau_1 \to \tau_2 \sim \tau_1' \to \tau_2'}{\Gamma \Vdash \mathsf{right}\,\gamma : \tau_2 \sim \tau_2'}
$$

$$
\frac{\text{Eq-All}}{\Gamma, \alpha \Vdash \gamma : \tau_1 \sim \tau_2}{\Gamma \Vdash \forall \alpha.\, \gamma : \forall \alpha.\, \tau_1 \sim \forall \alpha.\, \tau_2}
\qquad
\frac{\text{Eq-Inst}}{\Gamma \Vdash \gamma : \forall \alpha.\, \tau_1 \sim \forall \alpha.\, \tau_2 \qquad \Gamma \vdash \tau}{\Gamma \Vdash \gamma @ \tau : [\alpha \mapsto \tau]\tau_1 \sim [\alpha \mapsto \tau]\tau_2}
$$

Only equalities between *injective* type constructors can be decomposed.

Semantics

Coercions should be without computational content

?

Semantics

Coercions should be without computational content

▷ they are just type information, and should be erased at runtime

▷ they should not block redexes

▷ in Fc, we may always push them down inside terms, adding new reduction rules:

$$
\begin{array}{rcl}
(\gamma \lhd V_1)\, V_2 & \longrightarrow & \mathsf{right}\,\gamma \lhd (V_1\,(\mathsf{left}\,\gamma \lhd V_2)) \\
(\gamma \lhd V)\, \tau & \longrightarrow & (\gamma@\tau) \lhd (V\,\tau) \\
\gamma_1 \lhd (\gamma_2 \lhd V) & \longrightarrow & (\gamma_1;\gamma_2) \lhd V
\end{array}
$$

Semantics

Coercions should be without computational content

Always?

Semantics

Coercions should be without computational content

Except ...

?

Semantics

Coercions should be without computational content

Except for coercion abstractions that must stop the evaluation

Why?

Semantics

Coercions should be without computational content

Except for coercion abstractions that must stop the evaluation

▷ Otherwise, one could attempt to reduce M in $\lambda int \sim bool. M$
when M is *not* ($bool \lhd 0$), which is well-typed in this context.

▷ In call-by-value,

$$\lambda x : \tau_1 \sim \tau_2. M \quad \text{freezes} \quad \text{the evaluation of } M,$$
$$M \lhd \gamma \quad\quad\quad \text{resumes} \quad \text{the evaluation of } M.$$

Must always be enforced, even with other strategies

▷ Full reduction *at compile time*

?

Semantics

Coercions should be without computational content

Except for coercion abstractions that must stop the evaluation

 ▷ Otherwise, one could attempt to reduce M in $\lambda int \sim bool. M$
   when M is $not\,(bool \lhd 0)$, which is well-typed in this context.

 ▷ In call-by-value,

   $$\lambda x : \tau_1 \sim \tau_2. M \quad \text{freezes} \quad \text{the evaluation of } M,$$
   $$M \lhd \gamma \qquad\quad \text{resumes} \quad \text{the evaluation of } M.$$

   Must always be enforced, even with other strategies

 ▷ Full reduction *at compile time* may still be perfomed,

<div align="center">

?

</div>

Semantics

Coercions should be without computational content

Except for coercion abstractions that must stop the evaluation

> ▷ Otherwise, one could attempt to reduce M in $\lambda int \sim bool.\ M$
>   when M is *not* ($bool \lhd 0$), which is well-typed in this context.

> ▷ In call-by-value,

$$\lambda x : \tau_1 \sim \tau_2.\ M \quad \text{freezes} \quad \text{the evaluation of } M,$$
$$M \lhd \gamma \qquad\qquad \text{resumes} \quad \text{the evaluation of } M.$$

Must always be enforced, even with other strategies

> ▷ Full reduction *at compile time* may still be perfomed,
>   but be aware of stuck programs and treat them as dead branches.

Type soundness   *Syntactic proofs*

Type soundness

By subject reduction and progress with explicit coercions

Erasing semantics

Important and not so obvious.

$$\gamma \lhd M \quad \text{erases} \quad \text{to } M$$
$$\gamma \quad\quad\;\; \text{erases} \quad \text{to } \diamond$$

Slogan that "coercion have 0-bit information", *i.e.*
Coercions need not be passed at runtime—-but still block the reduction.
Expressions and typing rules.

COERCE
$$\frac{\Gamma \vdash M : \tau_1 \quad \Gamma \vdash \diamond : \tau_1 \sim \tau_2}{\Gamma \vdash M : \tau_2}$$

COERCION
$$\frac{\Gamma \Vdash \tau_1 \sim \tau_2}{\Gamma \vdash \diamond : \tau_1 \sim \tau_2}$$

COABS
$$\frac{\Gamma, x : \tau_1 \sim \tau_2 \vdash M : \tau}{\Gamma \vdash \lambda x : \tau_1 \sim \tau_2 . M : \tau_1 \sim \tau_2 \to \tau}$$

Type soundness   *Syntactic proofs*

The introduction of type equality constraints in System F has been
introduced and formalized by Sulzmann et al. [2007].

Scherer and Rémy [2015] show how strong reduction and confluence can
be recovered in the presence of possibly uninhabited coercions.

Type soundness   *Semantic proofs*

Equality coercions are a small logic of type conversions.

Type conversions may be enriched with more operations.

A very general form of coercions has been introduced by
Cretin and Rémy [2014].

The type soundness proof became too cumbersome to be conducted
syntactically.

Instead a semantic proof is used, interpreting types as sets of terms
(a technique similar to unary logical relations)

Type checking / inference

With explicit coercions, types are fully determined from expressions.

However, the user prefers to leave applications of Coerce implicit.

Then types becomes ambiguous: when leaving the scope of an equation:
which form should be used, among the equivalent ones?

This must be determined from the context, including the return type,
and calls for extra type annotations:

```
let rec eval : type a . a expr → a = fun x → match x with
  | Int x         → x   (* x : int, but a = int, should we return x : a? *)
  | Zerop x       → eval x > 0
  | If (b, e1, e2) → if eval b then eval e1 else eval e2
```

In ML, type annotations must be used to tell

- the type of the context
- which datatypes must be typed as GADTs.

In Coq, one must use return type annotations on matches.

Type inference in ML-like languages with GADTs

Simonet and Pottier [2007] gave a presentation of type inference for GADTs with general typing constraints for ML-like languages.

Pottier and Régis-Gianas [2006] introduced a stratified approach to better propagate constraints from outisde to inside GADTs contexts.

Vytiniotis et al. [2011] introduced the outside-in approach, used in Haskell, which restricts type information to flow from outside to inside GADT contexts.

Garrigue and Rémy [2013] introduced the notion of ambivalent types, used in OCaml, to restrict type occurrences that must be considered ambiguous and explicitly specified using type annotations.

Contents

- Algebraic Data Types
  - Equi- and iso- recursive types

- Existential types
  - Implicitly-type existential types passing
  - Iso-existential types

- Generalized Algebraic Datatypes

- Application to typed closure conversion
  - Environment passing
  - Closure passing

Type-preserving compilation

Compilation is type-preserving when each intermediate language is *explicitly typed*, and each compilation phase transforms a typed program into a typed program in the next intermediate language.

Why *preserve types* during compilation?

- it can help debug the compiler;
- types can be used to drive optimizations;
- types can be used to produce *proof-carrying code;*
- proving that types are preserved can be the first step towards proving that the *semantics* is preserved [Chlipala, 2007].

Type-preserving compilation

Type-preserving compilation exhibits an encoding of programming constructs into programming languages with usually richer type systems.

The encoding may sometimes be used directly as a programming idiom in the source language.

For example:

- Closure conversion requires an extension of the language with existential types, which happens to be very useful on their own.

- Closures are themselves a simple form of objects, which can also be explained with existential types.

- Defunctionalization may be done manually on some particular programs, *e.g.* in web applications to monitor the computation.

Type-preserving compilation

A classic paper by Morrisett *et al.* [1999] shows how to go from System F to Typed Assembly Language, while preserving types along the way. Its main passes are:

- *CPS conversion* fixes the order of evaluation, names intermediate computations, and makes all function calls tail calls;
- *closure conversion* makes environments and closures explicit, and produces a program where all functions are closed;
- allocation and initialization of tuples is made explicit;
- the calling convention is made explicit, and variables are replaced with (an unbounded number of) machine registers.

Translating types

In general, a type-preserving compilation phase involves not only a translation of *terms,* mapping M to $[\![M]\!]$, but also a translation of *types,* mapping τ to $[\![\tau]\!]$, with the property:

$$\Gamma \vdash M : \tau \quad \text{implies} \quad [\![\Gamma]\!] \vdash [\![M]\!] : [\![\tau]\!]$$

The translation of types carries a lot of information: examining it is often enough to guess what the translation of terms will be.

See the old lecture on type closure conversion.

Closure conversion

First-class functions may appear in the body of other functions. hence, their own body may contain free variables that will be bound to values during the evaluation in the execution environment.

Because they can be returned as values, and thus used outside of their definition environment, they must store their execution environment in their value.

A *closure* is the packaging of the code of a first-class function with its runtime environment, so that it becomes closed, *i.e.* independent of the runtime environment and can be moved and applied in another runtime environment.

Closures can also be used to represent recursive functions and objects (in the object-as-record-of-methods paradigm).

Source and target

In the following,

- the *source* calculus has *unary* λ-abstractions, which can have free variables;
- the *target* calculus has *binary* λ-abstractions, which must be *closed.*

Closure conversion can be easily extended to n-ary functions, or n-ary functions may be *uncurried* in a separate, type-preserving compilation pass.

Variants of closure conversion

There are at least two variants of closure conversion:

- in the *closure-passing variant,*
  the closure and the environment are a single memory block;

- in the *environment-passing variant,*
  the environment is a separate block, to which the closure points.

The impact of this choice on the translation of terms is minor.

Its impact on the translation of types is more important:
the closure-passing variant requires more type-theoretic machinery.

Closure-passing closure conversion

Let $\{x_1, \ldots, x_n\}$ be $\mathrm{fv}(\lambda x.\, a)$:

$$
\begin{aligned}
[\![\lambda x.\, a]\!] &= \text{ \textit{let} } code = \lambda(clo, x). \\
&\qquad \text{ \textit{let} } (\_, x_1, \ldots, x_n) = clo \text{ \textit{in} } [\![a]\!] \text{ \textit{in} } \\
&\qquad (code, x_1, \ldots, x_n) \\[1em]
[\![a_1\ a_2]\!] &= \text{ \textit{let} } clo = [\![a_1]\!] \text{ \textit{in} } \\
&\qquad \text{ \textit{let} } code = \mathsf{proj}_0\ clo \text{ \textit{in} } \\
&\qquad code\ (clo, [\![a_2]\!])
\end{aligned}
$$

(The variables $code$ and clo must be suitably fresh.)

Closure-passing closure conversion

Let $\{x_1, \ldots, x_n\}$ be $\mathrm{fv}(\lambda x.\, a)$:

$$
\begin{aligned}
[\![\lambda x.\, a]\!] \;=\; & \textit{let code} = \lambda(\mathit{clo}, x). \\
& \quad \textit{let } (\_, x_1, \ldots, x_n) = \mathit{clo} \textit{ in } [\![a]\!] \textit{ in} \\
& (\mathit{code}, x_1, \ldots, x_n) \\[4pt]
[\![a_1\ a_2]\!] \;=\; & \textit{let } \mathit{clo} = [\![a_1]\!] \textit{ in} \\
& \textit{let } \mathit{code} = \mathsf{proj}_0\ \mathit{clo} \textit{ in} \\
& \mathit{code}\ (\mathit{clo}, [\![a_2]\!])
\end{aligned}
$$

Important! The layout of the environment must be known only at the closure allocation site, not at the call site. In particular, $\mathsf{proj}_0\ \mathit{clo}$ need not know the size of clo.

Environment-passing closure conversion

Let $\{x_1, \ldots, x_n\}$ be $\mathrm{fv}(\lambda x.\, a)$:

$$
\begin{aligned}
[\![\lambda x.\, a]\!] &= \textit{let } code = \lambda(env, x). \\
&\qquad \textit{let } (x_1, \ldots, x_n) = env \textit{ in } [\![a]\!] \textit{ in} \\
&\quad (code, (x_1, \ldots, x_n)) \\[1em]
[\![a_1\ a_2]\!] &= \textit{let } (code, env) = [\![a_1]\!] \textit{ in} \\[0.5em]
&\quad code\ (env, [\![a_2]\!])
\end{aligned}
$$

Environment-passing closure conversion

Let $\{x_1, \ldots, x_n\}$ be $\mathrm{fv}(\lambda x.\, a)$:

$$
\begin{aligned}
[\![\lambda x.\, a]\!] \quad &= \quad \textit{let } code = \lambda(\mathit{env}, x). \\
&\qquad \textit{let } (x_1, \ldots, x_n) = \mathit{env} \textit{ in } [\![a]\!] \textit{ in} \\
&\qquad (\mathit{code}, (x_1, \ldots, x_n)) \\[2mm]
[\![a_1\ a_2]\!] \quad &= \quad \textit{let } (\mathit{code}, \mathit{env}) = [\![a_1]\!] \textit{ in} \\[2mm]
&\qquad \mathit{code}\ (\mathit{env}, [\![a_2]\!])
\end{aligned}
$$

Questions: How can closure conversion be made *type-preserving?*

Environment-passing closure conversion

Let $\{x_1, \ldots, x_n\}$ be $\mathrm{fv}(\lambda x.\, a)$:

$$
\begin{aligned}
[\![\lambda x.\, a]\!] \;=\; & \textit{let } code = \lambda(env, x). \\
& \quad \textit{let } (x_1, \ldots, x_n) = env \textit{ in } [\![a]\!] \textit{ in} \\
& (code, (x_1, \ldots, x_n))
\end{aligned}
$$

$$
\begin{aligned}
[\![a_1\ a_2]\!] \;=\; & \textit{let } (code, env) = [\![a_1]\!] \textit{ in} \\[4pt]
& code\ (env, [\![a_2]\!])
\end{aligned}
$$

Questions: How can closure conversion be made *type-preserving?*

The key issue is to find a sensible definition of the type translation.
In particular, what is the translation of a function type, $[\![\tau_1 \to \tau_2]\!]$?

Environment-passing closure conversion

Let $\{x_1, \ldots, x_n\}$ be $\mathrm{fv}(\lambda x.\, a)$:

$$
\begin{aligned}
[\![\lambda x.\, a]\!] \;=\; &\text{let } code = \lambda(env, x). \\
&\quad \text{let } (x_1, \ldots, x_n) = env \text{ in } [\![a]\!] \text{ in} \\
&(code, (x_1, \ldots, x_n))
\end{aligned}
$$

Assume $\Gamma \vdash \lambda x.\, a : \tau_1 \to \tau_2$.

Assume, *w.l.o.g.*. $\mathrm{dom}(\Gamma) = \mathrm{fv}(\lambda x.\, a) = \{x_1, \ldots, x_n\}$.

Write $[\![\Gamma]\!]$ for the tuple type $x_1 : [\![\tau_1']\!]; \ldots; x_n : [\![\tau_n']\!]$ where Γ is $x_1 : \tau_1'; \ldots; x_n : \tau_n'$. We also use $[\![\Gamma]\!]$ as a type to mean $[\![\tau_1']\!] \times \ldots \times [\![\tau_n']\!]$.

We have $\Gamma, x : \tau_1 \vdash a : \tau_2$, so in environment $[\![\Gamma]\!], x : [\![\tau_1]\!]$, we have

- env has type $[\![\Gamma]\!]$,
- $code$ has type $([\![\Gamma]\!] \times [\![\tau_1]\!]) \to [\![\tau_2]\!]$, and
- the entire closure has type $(([\![\Gamma]\!] \times [\![\tau_1]\!]) \to [\![\tau_2]\!]) \times [\![\Gamma]\!]$.

Now, *what should be the definition of $[\![\tau_1 \to \tau_2]\!]$?*

Towards a type translation

Can we adopt this as a definition?

$$[\![\tau_1 \to \tau_2]\!] \quad = \quad ((([\![\Gamma]\!] \times [\![\tau_1]\!]) \to [\![\tau_2]\!]) \times [\![\Gamma]\!]$$

Towards a type translation

Can we adopt this as a definition?

$$\llbracket \tau_1 \to \tau_2 \rrbracket \quad = \quad ((\llbracket \Gamma \rrbracket \times \llbracket \tau_1 \rrbracket) \to \llbracket \tau_2 \rrbracket) \times \llbracket \Gamma \rrbracket$$

Naturally not. This definition is mathematically ill-formed: we cannot use Γ out of the blue.

That is, this definition is not uniform: it depends on Γ, *i.e.* the size and layout of the environment.

Do we really need to have a uniform translation of types?

Towards a type translation

Yes, we do.

Towards a type translation

Yes, we do.

We need a uniform translation of types, not just because it is nice to have one, but because it describes a *uniform calling convention.*

If closures with distinct environment sizes or layouts receive distinct types, then we will be unable to translate this well-typed code:

Towards a type translation

Yes, we do.

We need a uniform translation of types, not just because it is nice to have one, but because it describes a *uniform calling convention.*

If closures with distinct environment sizes or layouts receive distinct types, then we will be unable to translate this well-typed code:

$$if \ldots then \ \lambda x. \, x + y \ else \ \lambda x. \, x$$

Furthermore, we want function invocations to be translated uniformly, without knowledge of the size and layout of the closure's environment.

Towards a type translation

Yes, we do.

We need a uniform translation of types, not just because it is nice to have one, but because it describes a *uniform calling convention.*

If closures with distinct environment sizes or layouts receive distinct types, then we will be unable to translate this well-typed code:

$$\text{if } \ldots \text{ then } \lambda x.\, x + y \text{ else } \lambda x.\, x$$

Furthermore, we want function invocations to be translated uniformly, without knowledge of the size and layout of the closure's environment.

So, *what could be the definition of $[\![\tau_1 \to \tau_2]\!]$?*

The type translation

The only sensible solution is:

$$[\![\tau_1 \to \tau_2]\!] = \exists\alpha.((\alpha \times [\![\tau_1]\!]) \to [\![\tau_2]\!]) \times \alpha$$

An *existential quantification* over the type of the environment abstracts away the differences in size and layout.

Enough information is retained to ensure that the application of the code to the environment is valid: this is expressed by letting the variable α occur twice on the right-hand side.

The type translation

The existential quantification also provides a form of *security*: the caller
cannot do anything with the environment except pass it as an argument
to the code; in particular, it cannot inspect or modify the environment.

For instance, in the source language, the following coding style
guarantees that x remains even, no matter how f is used:

$$\text{let } f = \text{let } x = \text{ref } 0 \text{ in } \lambda(). \, x := (x + 2); ! \, x$$

After closure conversion, the reference x is reachable via the closure of f.
A malicious, untyped client could write an odd value to x.
However, a *well-typed* client is unable to do so.

This encoding is not just type-preserving, but also *fully abstract:* it
preserves (a typed version of) observational
equivalence [Ahmed and Blume, 2008].

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Typed closure conversion

Everything is now set up to prove that, in System F with existential types:

$$\Gamma \vdash M : \tau \quad \text{implies} \quad [\![\Gamma]\!] \vdash [\![M]\!] : [\![\tau]\!]$$

Environment-passing closure conversion

Assume $\Gamma \vdash \lambda x.\, M : \tau_1 \to \tau_2$ and $\mathrm{dom}(\Gamma) = \{x_1, \ldots, x_n\} = \mathrm{fv}(\lambda x.\, M)$.

$$
\begin{aligned}
\llbracket \lambda x\!:\!\tau_1.\, M \rrbracket \;=\; & \mathit{let}\; \mathit{code} : \hspace{5cm} = \\
& \quad \lambda(\mathit{env}: \quad , x: \quad). \\
& \quad\quad \mathit{let}\; (x_1, \ldots, x_n : \quad) = \mathit{env}\; \mathit{in} \\
& \quad\quad \llbracket M \rrbracket \\
& \mathit{in} \\
& \mathit{pack} \quad , (\mathit{code}, (x_1, \ldots, x_n)) \\
& \mathit{as}
\end{aligned}
$$

Environment-passing closure conversion

Assume $\Gamma \vdash \lambda x.\, M : \tau_1 \to \tau_2$ and $\mathrm{dom}(\Gamma) = \{x_1, \ldots, x_n\} = \mathrm{fv}(\lambda x.\, M)$.

$$
\begin{aligned}
[\![\lambda x{:}\tau_1.\, M]\!] \quad = \quad & let\ code : \qquad\qquad\qquad\qquad\qquad = \\
& \quad \lambda(env : [\![\Gamma]\!], x : [\![\tau_1]\!]). \\
& \qquad let\ (x_1, \ldots, x_n : [\![\Gamma]\!]) = env\ in \\
& \qquad [\![M]\!] \\
& in \\
& pack \qquad , (code, (x_1, \ldots, x_n)) \\
& \quad as
\end{aligned}
$$

Environment-passing closure conversion

Assume $\Gamma \vdash \lambda x.\, M : \tau_1 \to \tau_2$ and $\mathrm{dom}(\Gamma) = \{x_1, \ldots, x_n\} = \mathrm{fv}(\lambda x.\, M)$.

$$\llbracket \lambda x{:}\tau_1.\, M \rrbracket = \begin{array}{l} \mathit{let\ code} : (\llbracket \Gamma \rrbracket \times \llbracket \tau_1 \rrbracket) \to \llbracket \tau_2 \rrbracket = \\ \quad \lambda(\mathit{env} : \llbracket \Gamma \rrbracket, x : \llbracket \tau_1 \rrbracket). \\ \quad\quad \mathit{let}\ (x_1, \ldots, x_n : \llbracket \Gamma \rrbracket) = \mathit{env\ in} \\ \quad\quad \llbracket M \rrbracket \\ \mathit{in} \\ \mathit{pack} \quad\ \ , (\mathit{code}, (x_1, \ldots, x_n)) \\ \quad \mathit{as} \end{array}$$

Environment-passing closure conversion

Assume $\Gamma \vdash \lambda x.\, M : \tau_1 \to \tau_2$ and $\mathrm{dom}(\Gamma) = \{x_1, \ldots, x_n\} = \mathrm{fv}(\lambda x.\, M)$.

$$
\begin{aligned}
[\![\lambda x\!:\!\tau_1.\, M]\!] \;=\; & \mathit{let}\; code : ([\![\Gamma]\!] \times [\![\tau_1]\!]) \to [\![\tau_2]\!] = \\
& \quad \lambda(\mathit{env} : [\![\Gamma]\!], x : [\![\tau_1]\!]).\\
& \quad\quad \mathit{let}\; (x_1, \ldots, x_n : [\![\Gamma]\!]) = \mathit{env}\; \mathit{in} \\
& \quad\quad [\![M]\!] \\
& \mathit{in} \\
& \mathit{pack}\; [\![\Gamma]\!], (code, (x_1, \ldots, x_n)) \\
& \mathit{as}\; \exists \alpha.((\alpha \times [\![\tau_1]\!]) \to [\![\tau_2]\!]) \times \alpha
\end{aligned}
$$

Environment-passing closure conversion

Assume $\Gamma \vdash \lambda x. M : \tau_1 \to \tau_2$ and $\mathrm{dom}(\Gamma) = \{x_1, \ldots, x_n\} = \mathrm{fv}(\lambda x. M)$.

$$
\begin{aligned}
[\![\lambda x{:}\tau_1.\, M]\!] \;=\;\; & \mathit{let}\; code : ([\![\Gamma]\!] \times [\![\tau_1]\!]) \to [\![\tau_2]\!] = \\
& \quad \lambda(\mathit{env} : [\![\Gamma]\!], x : [\![\tau_1]\!]). \\
& \qquad \mathit{let}\; (x_1, \ldots, x_n : [\![\Gamma]\!]) = \mathit{env}\; \mathit{in} \\
& \qquad [\![M]\!] \\
& \mathit{in} \\
& \mathit{pack}\; [\![\Gamma]\!], (code, (x_1, \ldots, x_n)) \\
& \mathit{as}\; \exists\alpha.(([\alpha] \times [\![\tau_1]\!]) \to [\![\tau_2]\!]) \times \alpha
\end{aligned}
$$

We find $[\![\Gamma]\!] \vdash [\![\lambda x{:}\tau_1.\, M]\!] : [\![\tau_1 \to \tau_2]\!]$, as desired.

Environment-passing closure conversion

Assume $\Gamma \vdash M : \tau_1 \to \tau_2$ and $\Gamma \vdash M_1 : \tau_1$.

$$\begin{aligned}
\llbracket M \ M_1 \rrbracket \ = \ & \textit{let } \alpha, (\textit{code} : (\alpha \times \llbracket \tau_1 \rrbracket) \to \llbracket \tau_2 \rrbracket, \textit{env} : \alpha) = \\
& \quad \textit{unpack } \llbracket M \rrbracket \textit{ in} \\
& \textit{code } (\textit{env}, \llbracket M_1 \rrbracket)
\end{aligned}$$

We find $\llbracket \Gamma \rrbracket \vdash \llbracket M \ M_1 \rrbracket : \llbracket \tau_2 \rrbracket$, as desired.

Environment-passing closure conversion   recursion

Recursive functions can be translated in this way, known as the "fix-code" variant [Morrisett and Harper, 1998] (leaving out type information):

$$
\begin{aligned}
[\![\mu f.\lambda x.M]\!] \;\; = \;\; & \textit{let rec } code\,(env, x) = \\
& \quad \textit{let } f = \textit{pack}\,(code, env) \textit{ in} \\
& \quad \textit{let } (x_1, \ldots, x_n) = env \textit{ in} \\
& \quad [\![M]\!] \textit{ in} \\
& \textit{pack}\,(code, (x_1, \ldots, x_n))
\end{aligned}
$$

where $\{x_1, \ldots, x_n\} = \mathrm{fv}(\mu f.\lambda x.M)$.

The translation of applications is unchanged: recursive and non-recursive functions have an identical calling convention.

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$$[\![M]\!]\textit{ in}$$
$$\textit{pack}\,(code, (x_1, \ldots, x_n))$$

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What is the weak point of this variant?

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& \quad \text{let } (x_1, \ldots, x_n) = env \text{ in} \\
& \quad \llbracket M \rrbracket \text{ in} \\
& \quad pack\,(code, (x_1, \ldots, x_n))
\end{aligned}
$$

where $\{x_1, \ldots, x_n\} = \mathrm{fv}(\mu f.\lambda x.M)$.

The translation of applications is unchanged: recursive and non-recursive functions have an identical calling convention.

What is the weak point of this variant?

A new closure is allocated at every call.

Environment-passing closure conversion   recursion

Instead, the "fix-pack" variant [Morrisett and Harper, 1998] uses an extra field in the environment to store a back pointer to the closure:

$$
\begin{aligned}
\llbracket \mu f.\lambda x.M \rrbracket \;=\; & \textit{let } code\,(env,x) = \\
& \quad \textit{let }(f, x_1, \ldots, x_n) = env \textit{ in} \\
& \quad \llbracket M \rrbracket \\
& \textit{in} \\
& \textit{let rec } clo = (code, (clo, x_1, \ldots, x_n)) \textit{ in} \\
& clo
\end{aligned}
$$

where $\{x_1, \ldots, x_n\} = \mathrm{fv}(\mu f.\lambda x.M)$.

This requires general, recursively-defined *values.* Closures are now *cyclic* data structures.

Environment-passing closure conversion   recursion

Here is how the "fix-pack" variant is type-checked. Assume
$\Gamma \vdash \mu f.\lambda x.M : \tau_1 \to \tau_2$ and $\mathrm{dom}(\Gamma) = \{x_1, \ldots, x_n\} = \mathrm{fv}(\mu f.\lambda x.M)$.

$$\llbracket \mu f \qquad .\lambda x.M \rrbracket =$$
     let code :   =
       $\lambda(env :$ $, x :$ $).$
          let $(f, x_1, \ldots, x_n) :$ $= env$ *in*
          $\llbracket M \rrbracket$ *in*
     let rec clo :   =
       pack $, (code, (clo, x_1, \ldots, x_n))$
        as
      in clo

Environment-passing closure conversion   recursion

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$$
\begin{aligned}
&\llbracket \mu f : \tau_1 \to \tau_2.\lambda x.M \rrbracket = \\
&\qquad \textit{let code} : \qquad\qquad\qquad\qquad\qquad\qquad\quad = \\
&\qquad\quad \lambda(\textit{env} : \llbracket f : \tau_1 \to \tau_2, \Gamma \rrbracket, x : \llbracket \tau_1 \rrbracket). \\
&\qquad\qquad \textit{let } (f, x_1, \ldots, x_n) : \llbracket f : \tau_1 \to \tau_2, \Gamma \rrbracket = \textit{env in} \\
&\qquad\qquad \llbracket M \rrbracket \textit{ in} \\
&\qquad \textit{let rec clo} : \qquad\qquad\qquad = \\
&\qquad\quad \textit{pack } \llbracket f : \tau_1 \to \tau_2, \Gamma \rrbracket, (\textit{code}, (\textit{clo}, x_1, \ldots, x_n)) \\
&\qquad\quad \textit{as} \\
&\qquad \textit{in } \textit{clo}
\end{aligned}
$$

Environment-passing closure conversion   recursion

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$$\llbracket \mu f : \tau_1 \to \tau_2.\lambda x.M \rrbracket =$$
        let $code : (\llbracket f : \tau_1 \to \tau_2; \Gamma \rrbracket \times \llbracket \tau_1 \rrbracket) \to \llbracket \tau_2 \rrbracket =$
          $\lambda(env : \llbracket f : \tau_1 \to \tau_2, \Gamma \rrbracket, x : \llbracket \tau_1 \rrbracket).$
            let $(f, x_1, \ldots, x_n) : \llbracket f : \tau_1 \to \tau_2, \Gamma \rrbracket = env$ *in*
            $\llbracket M \rrbracket$ *in*
        let rec $clo :$ $=$
          pack $\llbracket f : \tau_1 \to \tau_2, \Gamma \rrbracket, (code, (clo, x_1, \ldots, x_n))$
           as
        in clo

Environment-passing closure conversion   recursion

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$$\llbracket \mu f : \tau_1 \to \tau_2.\lambda x.M \rrbracket =$$
$$\quad \textit{let } code : (\llbracket f : \tau_1 \to \tau_2; \Gamma \rrbracket \times \llbracket \tau_1 \rrbracket) \to \llbracket \tau_2 \rrbracket =$$
$$\quad\quad \lambda(env : \llbracket f : \tau_1 \to \tau_2, \Gamma \rrbracket, x : \llbracket \tau_1 \rrbracket).$$
$$\quad\quad\quad \textit{let } (f, x_1, \ldots, x_n) : \llbracket f : \tau_1 \to \tau_2, \Gamma \rrbracket = env \textit{ in}$$
$$\quad\quad\quad \llbracket M \rrbracket \textit{ in}$$
$$\quad \textit{let rec } clo : \llbracket \tau_1 \to \tau_2 \rrbracket =$$
$$\quad\quad \textit{pack } \llbracket f : \tau_1 \to \tau_2, \Gamma \rrbracket, (code, (clo, x_1, \ldots, x_n))$$
$$\quad\quad\quad \textit{as}$$
$$\quad \textit{in } clo$$

Environment-passing closure conversion   recursion

Here is how the "fix-pack" variant is type-checked. Assume
$\Gamma \vdash \mu f. \lambda x. M : \tau_1 \to \tau_2$ and $\mathrm{dom}(\Gamma) = \{x_1, \ldots, x_n\} = \mathrm{fv}(\mu f. \lambda x. M)$.

$$\llbracket \mu f : \tau_1 \to \tau_2. \lambda x. M \rrbracket =$$
$$\textit{let code} : (\llbracket f : \tau_1 \to \tau_2; \Gamma \rrbracket \times \llbracket \tau_1 \rrbracket) \to \llbracket \tau_2 \rrbracket =$$
$$\lambda(\textit{env} : \llbracket f : \tau_1 \to \tau_2, \Gamma \rrbracket, x : \llbracket \tau_1 \rrbracket).$$
$$\textit{let } (f, x_1, \ldots, x_n) : \llbracket f : \tau_1 \to \tau_2, \Gamma \rrbracket = \textit{env in}$$
$$\llbracket M \rrbracket \textit{ in}$$
$$\textit{let rec clo} : \llbracket \tau_1 \to \tau_2 \rrbracket =$$
$$\textit{pack } \llbracket f : \tau_1 \to \tau_2, \Gamma \rrbracket, (\textit{code}, (\textit{clo}, x_1, \ldots, x_n))$$
$$\textit{as } \exists \alpha((\alpha \times \llbracket \tau_1 \rrbracket) \to \llbracket \tau_2 \rrbracket) \times \alpha)$$
$$\textit{in clo}$$

Environment-passing closure conversion   recursion

Here is how the "fix-pack" variant is type-checked. Assume
$\Gamma \vdash \mu f.\lambda x.M : \tau_1 \rightarrow \tau_2$ and $\mathrm{dom}(\Gamma) = \{x_1, \ldots, x_n\} = \mathrm{fv}(\mu f.\lambda x.M)$.

$$\llbracket \mu f : \tau_1 \rightarrow \tau_2.\lambda x.M \rrbracket =$$
$$\mathit{let}\ \mathit{code} : (\llbracket f : \tau_1 \rightarrow \tau_2; \Gamma \rrbracket \times \llbracket \tau_1 \rrbracket) \rightarrow \llbracket \tau_2 \rrbracket =$$
$$\lambda(\mathit{env} : \llbracket f : \tau_1 \rightarrow \tau_2, \Gamma \rrbracket, x : \llbracket \tau_1 \rrbracket).$$
$$\mathit{let}\ (f, x_1, \ldots, x_n) : \llbracket f : \tau_1 \rightarrow \tau_2, \Gamma \rrbracket = \mathit{env}\ \mathit{in}$$
$$\llbracket M \rrbracket\ \mathit{in}$$
$$\mathit{let}\ \mathit{rec}\ \mathit{clo} : \llbracket \tau_1 \rightarrow \tau_2 \rrbracket =$$
$$\mathit{pack}\ \llbracket f : \tau_1 \rightarrow \tau_2, \Gamma \rrbracket, (\mathit{code}, (\mathit{clo}, x_1, \ldots, x_n))$$
$$\mathit{as}\ \exists \alpha((\alpha \times \llbracket \tau_1 \rrbracket) \rightarrow \llbracket \tau_2 \rrbracket) \times \alpha)$$
$$\mathit{in}\ \mathit{clo}$$

Problem?

Environment-passing closure conversion   recursion

The recursive function may be polymorphic, but recursive calls are monomorphic...

We can generalize the encoding afterwards,

$$[\![\Lambda\vec{\beta}.\,\mu f : \tau_1 \to \tau_2.\lambda x.M]\!] = \Lambda\vec{\beta}.\,[\![\mu f : \tau_1 \to \tau_2.\lambda x.M]\!]$$

whenever the right-hand side is well-defined.

This allows the *indirect* compilation of polymorphic recursive functions as long as the recursion is monomorphic.

Fortunately, the encoding can be straightforwardly adapted to *directly* compile polymorphically recursive functions into polymorphic closure.

Environment-passing closure conversion   recursion

$$\llbracket \mu f : \forall \vec{\beta}. \tau_1 \to \tau_2. \lambda x.M \rrbracket =$$
$$\quad \textit{let code} : \forall \vec{\beta}. (\llbracket f : \forall \vec{\beta}. \tau_1 \to \tau_2; \Gamma \rrbracket \times \llbracket \tau_1 \rrbracket) \to \llbracket \tau_2 \rrbracket =$$
$$\quad\quad \lambda(\textit{env} : \llbracket f : \forall \vec{\beta}. \tau_1 \to \tau_2, \Gamma \rrbracket, x : \llbracket \tau_1 \rrbracket).$$
$$\quad\quad\quad \textit{let } (f, x_1, \ldots, x_n) : \llbracket f : \forall \vec{\beta}. \tau_1 \to \tau_2, \Gamma \rrbracket = \textit{env in}$$
$$\quad\quad\quad \llbracket M \rrbracket \textit{ in}$$
$$\quad \textit{let rec clo} : \llbracket \forall \vec{\beta}. \tau_1 \to \tau_2 \rrbracket =$$
$$\quad\quad \Lambda \vec{\beta}. \textit{pack } \llbracket f : \forall \vec{\beta}. \tau_1 \to \tau_2, \Gamma \rrbracket, (\textit{code } \vec{\beta}, (\textit{clo}, x_1, \ldots, x_n))$$
$$\quad\quad\quad \textit{as } \exists \alpha((\alpha \times \llbracket \tau_1 \rrbracket) \to \llbracket \tau_2 \rrbracket) \times \alpha)$$
$$\quad \textit{in clo}$$

The encoding is simple.

However, this requires the introduction of recursive non-functional values "let rec x = v". While this is a useful construct, it really alters the operational semantics and requires updating the type soundness proof.

- Algebraic Data Types
  - Equi- and iso- recursive types

- Existential types
  - Implicitly-type existential types passing
  - Iso-existential types

- Generalized Algebraic Datatypes

- Application to typed closure conversion
  - Environment passing
  - Closure passing

Closure-passing closure conversion

$$\llbracket \lambda x.\, M \rrbracket \;=\; \begin{aligned}[t] &\textit{let } code = \lambda(clo, x).\\ &\quad \textit{let } (\_, x_1, \ldots, x_n) = clo \textit{ in}\\ &\quad \llbracket M \rrbracket\\ &\textit{in } (code, x_1, \ldots, x_n) \end{aligned}$$

$$\llbracket M_1\, M_2 \rrbracket \;=\; \begin{aligned}[t] &\textit{let } clo = \llbracket M_1 \rrbracket \textit{ in}\\ &\textit{let } code = \mathsf{proj}_0\, clo \textit{ in}\\ &code\, (clo, \llbracket M_2 \rrbracket) \end{aligned}$$

where $\{x_1, \ldots, x_n\} = \text{fv}(\lambda x.\, M)$.

Closure-passing closure conversion

$$
\begin{aligned}
[\![\lambda x.\, M]\!] \;\; &= \;\; let\; code = \lambda(clo, x).\\
&\qquad\quad let\; (\_, x_1, \ldots, x_n) = clo \; in\\
&\qquad\quad [\![M]\!]\\
&\qquad in\; (\, code, x_1, \ldots, x_n)\\[1em]
[\![M_1\; M_2]\!] \;\; &= \;\; let\; clo = [\![M_1]\!] \; in\\
&\qquad let\; code = \mathsf{proj}_0\; clo \; in\\
&\qquad code\; (\, clo, [\![M_2]\!])
\end{aligned}
$$

where $\{x_1, \ldots, x_n\} = \mathrm{fv}(\lambda x.\, M)$.

How could we typecheck this? What are the difficulties?

Closure-passing closure conversion

$$\llbracket \lambda x.\, M \rrbracket = \begin{aligned}[t] &let\ code = \lambda(\,clo, x).\\ &\quad let\ (\_, x_1, \ldots, x_n) = clo\ in\\ &\quad \llbracket M \rrbracket\\ &in\ (\,code, x_1, \ldots, x_n) \end{aligned}$$

$$\llbracket M_1\ M_2 \rrbracket = \begin{aligned}[t] &let\ clo = \llbracket M_1 \rrbracket\ in\\ &let\ code = \mathsf{proj}_0\ clo\ in\\ &code\ (\,clo, \llbracket M_2 \rrbracket) \end{aligned}$$

There are two difficulties:

- a closure is a tuple, whose *first* field should be *exposed* (it is the code pointer), while the number and types of the remaining fields should be abstract;
- the first field of the closure contains a function that expects *the closure itself* as its first argument.

Closure-passing closure conversion

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What type-theoretic mechanisms could we use to describe this?

Closure-passing closure conversion

There are two difficulties:

- a closure is a tuple, whose *first* field should be *exposed* (it is the code pointer), while the number and types of the remaining fields should be abstract;

- the first field of the closure contains a function that expects *the closure itself* as its first argument.

What type-theoretic mechanisms could we use to describe this?

- existential quantification over the *tail* of a tuple (a.k.a. a *row*);
- *recursive types*.

Tuples, rows, row variables

The standard tuple types that we have used so far are:

$$\tau \quad ::= \quad \dots \mid \Pi\, R \qquad \text{– types}$$
$$R \quad ::= \quad \epsilon \mid (\tau; R) \qquad \text{– rows}$$

The notation $(\tau_1 \times \dots \times \tau_n)$ was sugar for $\Pi\, (\tau_1; \dots; \tau_n; \epsilon)$.

Let us now introduce *row variables* and allow *quantification* over them:

$$\tau \quad ::= \quad \dots \mid \Pi\, R \mid \forall \rho.\, \tau \mid \exists \rho.\tau \qquad \text{– types}$$
$$R \quad ::= \quad \rho \mid \epsilon \mid (\tau; R) \qquad \text{– rows}$$

This allows reasoning about the first few fields of a tuple whose length is not known.

Typing rules for tuples

The typing rules for tuple construction and deconstruction are:

$$
\begin{array}{cc}
\text{TUPLE} & \text{PROJ} \\
\dfrac{\forall i. \in [1,n] \quad \Gamma \vdash M_i : \tau_i}{\Gamma \vdash (M_1, \ldots, M_n) : \Pi\,(\tau_1; \ldots; \tau_n; \epsilon)} & \dfrac{\Gamma \vdash M : \Pi\,(\tau_1; \ldots; \tau_i; R)}{\Gamma \vdash \text{proj}_i\, M : \tau_i}
\end{array}
$$

These rules make sense with or without row variables

Projection does not care about the fields beyond i. Thanks to row variables, this can be expressed in terms of *parametric polymorphism:*

$$
\text{proj}_i : \forall \alpha. {}_1 \ldots \alpha_i \rho.\ \Pi\,(\alpha_1; \ldots; \alpha_i; \rho) \to \alpha_i
$$

About Rows

Rows were invented by Wand and improved by Rémy in order to ascribe precise types to operations on *records.*

The case of tuples, presented here, is simpler.

Rows are used to describe *objects* in Objective Caml [Rémy and Vouillon, 1998].

Rows are explained in depth by Pottier and Rémy [Pottier and Rémy, 2005].

Closure-passing closure conversion

Rows and recursive types allow to define the translation of types in the closure-passing variant:

$$
\begin{aligned}
& [\![\tau_1 \to \tau_2]\!] \\
= \ & \exists\rho. && \rho \text{ describes the environment} \\
& \quad \mu\alpha. && \alpha \text{ is the concrete type of the closure} \\
& \quad\quad \Pi\ (&& \text{a tuple...} \\
& \quad\quad\quad (\alpha \times [\![\tau_1]\!]) \to [\![\tau_2]\!]; && \text{...that begins with a code pointer...} \\
& \quad\quad\quad \rho && \text{...and continues with the environment} \\
& \quad\quad)
\end{aligned}
$$

See Morrisett and Harper's "fix-type" encoding [1998].

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Question: Why is it $\exists\rho.\ \mu\alpha.\ \tau$ and not $\mu\alpha.\ \exists\rho.\ \tau$

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\end{aligned}
$$

See Morrisett and Harper's "fix-type" encoding [1998].

Question: Why is it $\exists \rho.\ \mu\alpha.\ \tau$ and not $\mu\alpha.\ \exists \rho.\ \tau$

The type of the environment is fixed once for all and does not change at each recursive call.

Closure-passing closure conversion

Rows and recursive types allow to define the translation of types in the closure-passing variant:

$$
\begin{aligned}
&\quad [\![\tau_1 \to \tau_2]\!] \\
&= \exists\rho. && \rho \text{ describes the environment} \\
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&\quad\quad\quad \Pi\,(&& \text{a tuple...} \\
&\quad\quad\quad\quad (\alpha \times [\![\tau_1]\!]) \to [\![\tau_2]\!]; && \text{...that begins with a code pointer...} \\
&\quad\quad\quad\quad \rho && \text{...and continues with the environment} \\
&\quad\quad\quad)
\end{aligned}
$$

See Morrisett and Harper's "fix-type" encoding [1998].

Question: Notice that ρ appears only once. Any comments?

Closure-passing closure conversion

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Usually, an existential type variable appears both at positive and negative occurrences.

Closure-passing closure conversion

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&\quad\quad \Pi\ (&& \text{a tuple...} \\
&\quad\quad\quad (\alpha \times \llbracket \tau_1 \rrbracket) \to \llbracket \tau_2 \rrbracket; && \text{...that begins with a code pointer...} \\
&\quad\quad\quad \rho && \text{...and continues with the environment} \\
&\quad\quad)
\end{aligned}
$$

See Morrisett and Harper's "fix-type" encoding [1998].

Question: Notice that ρ appears only once. Any comments?

Usually, an existential type variable appears both at positive and negative occurrences. Here, the variable appear only at a negative occurrence, but in a recursive part of the type that can be unfolded.

Closure-passing closure conversion

Let $Clo(R)$ abbreviate $\mu\alpha.\Pi\,((\alpha \times [\![\tau_1]\!]) \to [\![\tau_2]\!]; R)$.

Let $UClo(R)$ abbreviate its unfolded version,
$\Pi\,((Clo(R) \times [\![\tau_1]\!]) \to [\![\tau_2]\!]; R)$.

We have $[\![\tau_1 \to \tau_2]\!] = \exists\rho.Clo(\rho)$.

$$[\![\lambda x\colon\quad.\,M]\!] = \text{let } code : \qquad\qquad\qquad\qquad =$$
$$\lambda(clo:\qquad\quad, x:\quad).$$
$$\text{let }(\_, x_1, \ldots, x_n):\qquad\qquad = \text{unfold } clo \text{ in}$$
$$[\![M]\!] \text{ in}$$
$$pack\quad,(fold\,(code, x_1, \ldots, x_n))$$
$$\text{as}$$

$$[\![M_1\ M_2]\!] = \text{let } \rho, clo = \text{unpack } [\![M_1]\!] \text{ in}$$
$$\text{let } code :\qquad\qquad\qquad\qquad =$$
$$proj_0\,(\text{unfold } clo) \text{ in}$$
$$code\,(clo, [\![M_2]\!])$$

Closure-passing closure conversion

Let $Clo(R)$ abbreviate $\mu\alpha.\Pi\,((\alpha \times [\![\tau_1]\!]) \to [\![\tau_2]\!]; R)$.

Let $UClo(R)$ abbreviate its unfolded version,
$\Pi\,((Clo(R) \times [\![\tau_1]\!]) \to [\![\tau_2]\!]; R)$.

We have $[\![\tau_1 \to \tau_2]\!] = \exists\rho.Clo(\rho)$.

$$
\begin{aligned}
[\![\lambda x\!:\![\![\tau_1]\!].\,M]\!] \;=\;\; & \textit{let code} : (Clo([\![\Gamma]\!]) \times [\![\tau_1]\!]) \to [\![\tau_2]\!] = \\
& \quad \lambda(\,clo : Clo([\![\Gamma]\!], x : [\![\tau_1]\!]). \\
& \qquad \textit{let}\,(\_, x_1, \ldots, x_n) : UClo[\![\Gamma]\!] = \textit{unfold clo in} \\
& \qquad [\![M]\!] \textit{ in} \\
& \textit{pack}\,[\![\Gamma]\!], (\textit{fold}\,(code, x_1, \ldots, x_n)) \\
& \textit{as}\,\exists\rho.\,Clo(\rho)
\end{aligned}
$$

$$
\begin{aligned}
[\![M_1\,M_2]\!] \;=\;\; & \textit{let}\,\rho, clo = \textit{unpack}\,[\![M_1]\!]\,\textit{in} \\
& \textit{let code} : (Clo(\rho) \times [\![\tau_1]\!]) \to [\![\tau_2]\!] = \\
& \quad \textit{proj}_0\,(\textit{unfold clo})\,\textit{in} \\
& code\,(clo, [\![M_2]\!])
\end{aligned}
$$

Closure-passing closure conversion   recursive functions

In the closure-passing variant, recursive functions can be translated as:

$$\llbracket \mu f.\lambda x.M \rrbracket = \text{\textit{let} } code = \lambda(clo, x).$$
$$\text{\textit{let} } f = clo \text{ \textit{in}}$$
$$\text{\textit{let} } (\_, x_1, \ldots, x_n) = clo \text{ \textit{in}}$$
$$\llbracket M \rrbracket$$
$$\text{\textit{in} } (code, x_1, \ldots, x_n)$$

where $\{x_1, \ldots, x_n\} = \text{fv}(\mu f.\lambda x.M)$.

No extra field or extra work is required to store or construct a representation of the free variable f: the closure itself plays this role.

However, this untyped code can only be typechecked when recursion is monomorphic.

Exercise:

Check well-typedness with monomorphic recursion.

Closure-passing closure conversion   recursive functions

The problem to adapt this encoding to polymorphic recursion is that recursive occurrences of f are rebuilt from the current invocation of the closure, *i.e.* is monomorphic since the closure is invoked after type specialization.

By contrast, in the environment passing encoding, the environment contained a polymorphic binding for the recursive calls that was filled with the closure before its invokation, *i.e.* with a polymorphic type.

Fortunately, we may slightly change the encoding, using a recursive closure as in the type-passing version, to allow typechecking in System F.

Closure-passing closure conversion   recursive functions

Let τ be $\forall \vec{\alpha}. \tau_1 \to \tau_2$ and Γ_f be $f : \tau, \Gamma$ where $\vec{\beta} \# \Gamma$

$$\llbracket \mu f : \tau. \lambda x.M \rrbracket = \text{let } code =$$
$$\Lambda \vec{\beta}. \lambda(clo : Clo\llbracket \Gamma_f \rrbracket, x : \llbracket \tau_1 \rrbracket).$$
$$\text{let } (\_code, f, x_1, \ldots, x_n) : \forall \vec{\beta}. UClo(\llbracket \Gamma_f \rrbracket) =$$
$$\text{unfold } clo \text{ in}$$
$$\llbracket M \rrbracket \text{ in}$$
$$\text{let rec } clo : \forall \vec{\beta}. \exists \rho. Clo(\rho) = \Lambda \vec{\beta}.$$
$$\text{pack } \llbracket \Gamma \rrbracket, (\text{fold } (code \ \vec{\beta}, clo, x_1, \ldots, x_n)) \text{ as } \exists \rho. Clo(\rho)$$
$$\text{in } clo$$

Remind that $Clo(R)$ abbreviates $\mu\alpha.\Pi \left((\alpha \times \llbracket \tau_1 \rrbracket) \to \llbracket \tau_2 \rrbracket; R \right)$. Hence, $\vec{\beta}$ are free variables of $Clo(R)$.

Here, a polymorphic recursive function is *directly* compiled into a polymorphic recursive closure. Notice that the type of closures is unchanged so the encoding of applications is also unchanged.

Mutually recursive functions   Environment passing

Can we compile mutually recursive functions?

$$M \triangleq \mu(f_1, f_2).(\lambda x_1. M_1, \lambda x_2. M_2)$$

Environment passing:

$$\llbracket M \rrbracket \quad =$$

Mutually recursive functions   Environment passing

Can we compile mutually recursive functions?

$$M \triangleq \mu(f_1, f_2).(\lambda x_1. M_1, \lambda x_2. M_2)$$

Environment passing:

$$
\begin{aligned}
\llbracket M \rrbracket \; = \; & \text{let } code_i = \lambda(env, x). \\
& \quad \text{let } (f_1, f_2, x_1, \ldots, x_n) = env \text{ in} \\
& \quad \llbracket M_i \rrbracket \\
& \text{in} \\
& \text{let rec } clo_1 = (code_1, (clo_1, clo_2, x_1, \ldots, x_n)) \\
& \quad \text{and } clo_2 = (code_2, (clo_1, clo_2, x_1, \ldots, x_n)) \text{ in} \\
& clo_1, clo_2
\end{aligned}
$$

Mutually recursive functions   Environment passing

Can we compile mutually recursive functions?

$$M \triangleq \mu(f_1, f_2).(\lambda x_1.\, M_1, \lambda x_2.\, M_2)$$

Environment passing:

$$
\begin{aligned}
\llbracket M \rrbracket \;=\; & \textit{let } code_i = \lambda(env, x). \\
& \quad \textit{let } (f_1, f_2, x_1, \ldots, x_n) = env \textit{ in} \\
& \quad \llbracket M_i \rrbracket \\
& \textit{in} \\
& \textit{let rec } clo_1 = (code_1, (clo_1, clo_2, x_1, \ldots, x_n)) \\
& \quad \textit{and } clo_2 = (code_2, (clo_1, clo_2, x_1, \ldots, x_n)) \textit{ in} \\
& clo_1, clo_2
\end{aligned}
$$

Comments?

Mutually recursive functions   Environment passing

Can we compile mutually recursive functions?

$$M \triangleq \mu(f_1, f_2).(\lambda x_1. M_1, \lambda x_2. M_2)$$

Environment passing:

$$
\begin{aligned}
\llbracket M \rrbracket \; = \; & \textit{let } code_i = \lambda(env, x). \\
& \quad \textit{let } (f_1, f_2, x_1, \ldots, x_n) = env \textit{ in} \\
& \quad \llbracket M_i \rrbracket \\
& \textit{in} \\
& \textit{let rec } env = (clo_1, clo_2, x_1, \ldots, x_n) \\
& \quad \textit{and } clo_1 = (code_1, env) \\
& \quad \textit{and } clo_2 = (code_2, env) \textit{ in} \\
& clo_1, clo_2
\end{aligned}
$$

Mutually recursive functions   Closure passing

Can we compile mutually recursive functions?

$$M \triangleq \mu(f_1, f_2).(\lambda x_1. M_1, \lambda x_2. M_2)$$

Closure passing:

> let $code_i = \lambda(clo, x)$.
>   let $(\_, f_1, f_2, x_1, \ldots, x_n) = clo$ in $[\![M_i]\!]$
> in
> let rec $clo_1 = (code_1, clo_1, clo_2, x_1, \ldots, x_n)$
>   and $clo_2 = (code_2, clo_1, clo_2, x_1, \ldots, x_n)$
> in clo_1, clo_2

Mutually recursive functions   Closure passing

Can we compile mutually recursive functions?

$$M \triangleq \mu(f_1, f_2).(\lambda x_1. M_1, \lambda x_2. M_2)$$

Closure passing:

$$
\begin{aligned}
&\textit{let } code_i = \lambda(clo, x). \\
&\quad \textit{let } (\_, f_1, f_2, x_1, \ldots, x_n) = clo \textit{ in } [\![M_i]\!] \\
&\textit{in} \\
&\textit{let rec } clo_1 = (code_1, clo_1, clo_2, x_1, \ldots, x_n) \\
&\quad \textit{and } clo_2 = (code_2, clo_1, clo_2, x_1, \ldots, x_n) \\
&\textit{in } clo_1, clo_2
\end{aligned}
$$

Question: Can we share the closures c_1 and c_2 in case n is large?

Mutually recursive functions   Closure passing

Can we compile mutually recursive functions?

$$M \triangleq \mu(f_1, f_2).(\lambda x_1. M_1, \lambda x_2. M_2)$$

Closure passing:

> let $code_1 = \lambda(clo, x)$.
>   let $(\_code_1, \_code_2, f_1, f_2, x_1, \ldots, x_n) = clo$ in $[\![M_1]\!]$ in
> let $code_2 = \lambda(clo, x)$.
>   let $(\_code_2, f_1, f_2, x_1, \ldots, x_n) = clo$ in $[\![M_2]\!]$ in
> let rec $clo_1 = (code_1, code_2, clo_1, clo_2, x_1, \ldots, x_n)$ and $clo_2 = clo_1.tail$
>   in clo_1, clo_2

- $clo_1.tail$ returns a pointer to the tail $(code_2, clo_1, clo_2, x_1, \ldots, x_n)$ of clo_1 without allocating a new tuple.
- This is only possible with some support from the GC (and extra-complexity and runtime cost for GC)

Optimizing representations

Can closure passing and environment passing be mixed?

Encoding of objects

The closure-passing representation of mutually recursive functions is similar to the representations of objects in the object-as-record-of-functions paradigm:

A class definition is an object generator:

$$
\begin{aligned}
&class\ c\ (x_1, \ldots x_q)\{ \\
&\quad \textit{meth}\ m_1 = M_1 \\
&\quad \ldots \\
&\quad \textit{meth}\ m_p = M_p \\
&\}
\end{aligned}
$$

Given arguments for parameter $x_1, \ldots x_1$, it will build recursive methods $m_1, \ldots m_n$.

Encoding of objects

A class can be compiled into an object closure:

$$
\begin{aligned}
let\ m = \\
\quad let\ m_1 = \lambda(m, x_1, \ldots, x_q).\, M_1\ in \\
\quad \ldots \\
\quad let\ m_p = \lambda(m, x_1, \ldots, x_q).\, M_p\ in \\
\quad \{m_1, \ldots, m_p\}\ in \\
\lambda x_1 \ldots x_q.\, (m, x_1, \ldots x_q)
\end{aligned}
$$

Each m_i is bound to the code for the corresponding method.
The code of all methods are combined into a record of methods,
which is shared between all objects of the same class.

Calling method m_i of an object p is

$$(proj_0\ p).m_i\ p$$

How can we type the encoding?

Typed encoding of objects

Let τ_i be the type of M_i, and row R describe the types of $(x_1, \ldots x_q)$.

Let $Clo(R)$ be $\mu\alpha.\Pi(\{(m_i : \alpha \to \tau_i)^{i\in 1..n}\}; R)$ and $UClo(R)$ its unfolding.

Fields R are hidden in an existential type $\exists\rho.\,\mu\alpha.\Pi(\{(m_i : \alpha \to \tau_i)^{i\in I}\}; \rho)$:

$$
\begin{aligned}
&let\ m = \{ \\
&\quad m_1 = \lambda(m, x_1, \ldots x_q : UClo(R)).\, [\![M_1]\!] \\
&\quad \ldots \\
&\quad m_p = \lambda(m, x_1, \ldots x_q : UClo(R)).\, [\![M_p]\!] \\
&\} \ in \\
&\lambda x_1.\, \ldots \lambda x_q.\, pack\ R, fold\ (m, x_1, \ldots x_q)\ as\ \exists\rho.\,(M, \rho)
\end{aligned}
$$

Calling a method of an object p of type M is

$$p \# m_i \triangleq let\ \rho, z = unpack\ p\ in\ (proj_0\ unfold\ z).m_i\ z$$

An object has a recursive type but it is *not* a recursive value.

Typed encoding of objects

Typed encoding of objects were first studied in the 90's to understand what objects really are in a type setting.

These encodings are in fact type-preserving compilation of (primitive) objects.

There are several variations on these encodings. See [Bruce et al., 1999] for a comparison.

See [Rémy, 1994] for an encoding of objects in (a small extension of) ML with iso-existentials and universals.

See [Abadi and Cardelli, 1996, 1995] for more details on primitive objects.

Moral of the story

Type-preserving compilation is rather *fun.* (Yes, really!)

It forces compiler writers to make the structure of the compiled program *fully explicit,* in type-theoretic terms.

In practice, building explicit type derivations, ensuring that they remain small and can be efficiently typechecked, can be a lot of work.

Optimizations

Because we have focused on type preservation, we have studied only naïve closure conversion algorithms.

More ambitious versions of closure conversion require program analysis: see, for instance, Steckler and Wand [1997]. These versions *can* be made type-preserving.

Other challenges

Defunctionalization, an alternative to closure conversion, offers an interesting challenge, with a simple solution [Pottier and Gauthier, 2006].

Designing an efficient, type-preserving compiler for an *object-oriented language* is quite challenging. See, for instance, Chen and Tarditi [2005].

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(Most titles have a clickable mark "▷" that links to online versions.)

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