

Liberating effects with rows and handlers

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March 31st, 2016

(joint work with Daniel Hillerström)

Motivation

A framework for modular programming with effects

- ▶ starting point: abstract computations over signatures of effectful operations $\{op_i : A_i \rightarrow B_i\}_i$
- ▶ modularity
 - ▶ interpreters for abstract computations defined in terms of interpretations of the underlying effectful operations
 - ▶ multiple interpreters
 - ▶ composable interpreters
- ▶ an *effect handler* is an interpreter for abstract computations
 - ▶ a *closed* effect handler interprets a fixed set of operations
 - ▶ an *open* effect handler interprets a fixed set of operations, and generically forwards all others (crucial for composition)





Abstract computations as trees

An abstract computation of type $\text{Comp } E A$ over effect signature $E = \{op_i : A_i \rightarrow B_i\}_i$ with return type A is a tree where

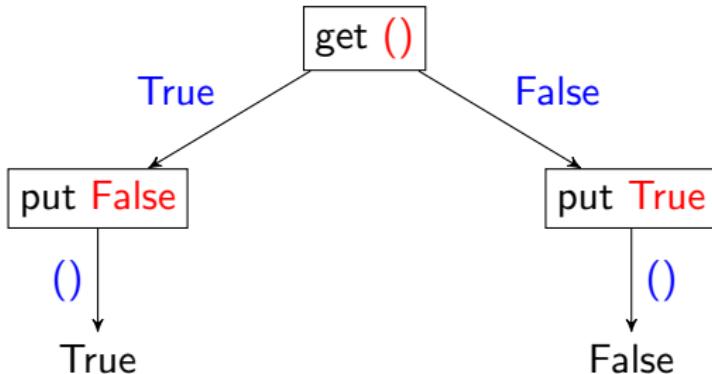
- ▶ nodes are labelled with operations and operation parameters
- ▶ edges are labelled with operation result values
- ▶ leaves are labelled with final return values of type A

Algebraic effects [Plotkin and Power, 2001]: same story, but trees are quotiented by equations

Example: bit toggling

$E = \{\text{get} : 1 \rightarrow \text{Bool}$
 $\quad \text{put} : \text{Bool} \rightarrow 1\}$
 $A = \text{Bool}$

toggle =
let $x \leftarrow \text{get}()$ **in**
 $\text{put}(\neg x); x$

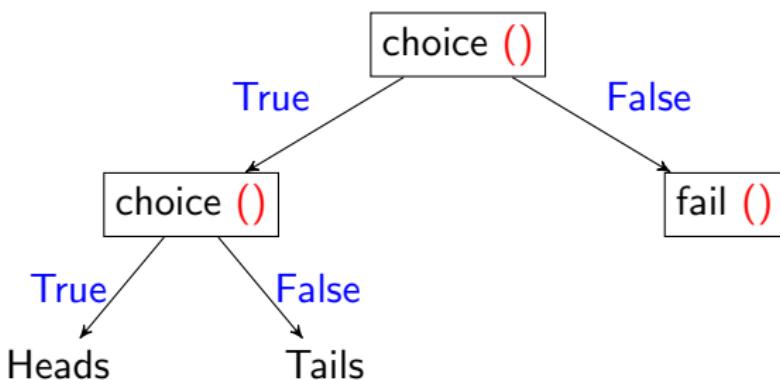


Example: drunk coin toss

$E = \{\text{choice} : 1 \rightarrow \text{Bool}$
 $\quad \text{fail} : 1 \rightarrow 0\}$

$A = \text{Heads} + \text{Tails}$

```
drunkToss =  
  if choice() then  
    if choice() then Heads  
    else Tails  
  else fail()
```



Effect handlers (Pretnar and Plotkin)

An effect handler is an interpreter for abstract computations of type

$\text{Comp } E A \rightarrow B$

for some target type B (which may itself be an abstract computation over a different signature).

An effect handler is defined as a fold over a computation tree, specifying how return values and operations are interpreted.

return $x \mapsto M$

$op_1 \ p \ k \mapsto N_1$

...

$op_n \ p \ k \mapsto N_n$

In each N_i , the variable k is bound to a function for invoking H on the subtrees of the current node.

Example: closed state handlers

State $S = \{\text{get} : 1 \rightarrow S, \text{put} : S \rightarrow 1\}$

$\text{evalState} : \text{Comp } (\text{State } S) A \rightarrow (S \rightarrow A)$

$\text{evalState} = \text{return } x \mapsto \lambda s. x$

get $() k$	$\mapsto \lambda s. k \ s \ s$	$(k: S \rightarrow (S \rightarrow A))$
put $t k$	$\mapsto \lambda s. k () t$	$(k: 1 \rightarrow (S \rightarrow A))$

$\text{logState} : \text{Comp } (\text{State } S) A \rightarrow (S \rightarrow A \times \text{List } S)$

$\text{logState} = \text{return } x \mapsto \lambda s. (x, [s])$

get $() k$	$\mapsto \lambda s. k \ s \ s$
put $t k$	$\mapsto \lambda s. \text{let } (x, ss) \leftarrow k () t \text{ in }$ $(x, s :: ss)$

$(\text{handle toggle with evalState}) \text{ true} = \text{true}$

$(\text{handle toggle with logState}) \text{ true} = (\text{true}, [\text{true}, \text{false}])$

Composing handlers

- ▶ *closed handlers* only handle the operations explicitly listed in the operation clauses
- ▶ *open handlers*, in addition to handling the explicitly specified operations, also *forward* all other operations
- ▶ open handlers support composition

Links

- ▶ statically-typed functional programming language for the web [Cooper, Lindley, Wadler, Yallop, 2006]
- ▶ relevant features: call-by-value, type inference, first-class continuations, row types (for records, variants, session types, and **effects**)

Effect handler implementation

- ▶ adapts existing infrastructure for row-based effects
- ▶ adapts existing infrastructure for first-class continuations

(Links demo)

Kinding rules

<p>TYVAR</p> $\frac{}{\Delta, \alpha : K \vdash \alpha : K}$	<p>COMP</p> $\frac{\Delta \vdash A : \text{Type} \quad \Delta \vdash E : \text{Effect}}{\Delta \vdash A!E : \text{Comp}}$
<p>FUN</p> $\frac{\Delta \vdash A : \text{Type} \quad \Delta \vdash C : \text{Comp}}{\Delta \vdash A \rightarrow C : \text{Type}}$	<p>FORALL</p> $\frac{\Delta, \alpha : K \vdash C : \text{Comp}}{\Delta \vdash \forall \alpha^K.C : \text{Type}}$
<p>RECORD</p> $\frac{\Delta \vdash R : \text{Row}_\emptyset}{\Delta \vdash \langle R \rangle : \text{Type}}$	<p>VARIANT</p> $\frac{\Delta \vdash R : \text{Row}_\emptyset}{\Delta \vdash [R] : \text{Type}}$
<p>EMPTYROW</p> $\frac{}{\Delta \vdash \cdot : \text{Row}_{\mathcal{L}}}$	<p>EFFECT</p> $\frac{\Delta \vdash R : \text{Row}_\emptyset}{\Delta \vdash \{R\} : \text{Effect}}$
<p>PRESENT</p> $\frac{\Delta \vdash A : \text{Type}}{\Delta \vdash \text{Pre}(A) : \text{Presence}}$	<p>EXTENDROW</p> $\frac{\Delta, P : \text{Presence} \quad \Delta, R : \text{Row}_{\mathcal{L} \uplus \{\ell\}}}{\Delta \vdash \ell : P; R : \text{Row}_{\mathcal{L}}}$
<p>ABSENT</p> $\frac{}{\Delta \vdash \text{Abs} : \text{Presence}}$	

Value typing rules

$$\frac{\text{T-VAR}}{x : A \in \Gamma} \quad \frac{}{\Delta ; \Gamma \vdash x : A}$$

$$\frac{\text{T-LAM}}{\Delta ; \Gamma, x : A \vdash M : C} \quad \frac{}{\Delta ; \Gamma \vdash \lambda x^A . M : A \rightarrow C}$$

$$\frac{\text{T-POLYLAM}}{\Delta, \alpha : K ; \Gamma \vdash M : C \quad \alpha \notin FTV(\Gamma)} \quad \frac{}{\Delta ; \Gamma \vdash \Lambda \alpha^K . M : \forall \alpha^K . C}$$

$$\frac{\text{T-UNIT}}{\Delta ; \Gamma \vdash \langle \rangle : \langle \rangle}$$

$$\frac{\text{T-EXTEND}}{\Delta ; \Gamma \vdash V : A \quad \Delta ; \Gamma \vdash W : \langle \ell : \text{Abs} ; R \rangle} \quad \frac{}{\Delta ; \Gamma \vdash \langle \ell = V ; W \rangle : \langle \ell : \text{Pre}(A) ; R \rangle}$$

$$\frac{\text{T-INJECT}}{\Delta ; \Gamma \vdash V : A} \quad \frac{}{\Delta ; \Gamma \vdash (\ell V)^R : [\ell : \text{Pre}(A) ; R]}$$

Computation typing rules

T-APP

$$\frac{\Delta; \Gamma \vdash V : A \rightarrow C \quad \Delta; \Gamma \vdash W : B}{\Delta; \Gamma \vdash V W : C}$$

T-POLYAPP

$$\frac{\Delta; \Gamma \vdash V : \forall \alpha^K. C \quad \Delta \vdash A : K}{\Delta; \Gamma \vdash VA : C[A/\alpha]}$$

T-SPLIT

$$\frac{\Delta; \Gamma \vdash V : \langle \ell : \text{Pre}(A); R \rangle \quad \Delta; \Gamma, x : A, y : \langle \ell : \text{Abs}; R \rangle \vdash N : C}{\Delta; \Gamma \vdash \mathbf{let} \langle \ell = x; y \rangle \leftarrow V \mathbf{in} N : C}$$

T-CASE

$$\frac{\Delta; \Gamma \vdash V : [\ell : \text{Pre}(A); R] \quad \Delta; \Gamma, x : A \vdash M : C \quad \Delta; \Gamma, y : [\ell : \text{Abs}; R] \vdash N : C}{\Delta; \Gamma \vdash \mathbf{case} \ V \{ \ell \ x \mapsto M; y \mapsto N \} : C}$$

T-ABSURD

$$\frac{\Delta; \Gamma \vdash V : []}{\Delta; \Gamma \vdash \mathbf{absurd}^A \ V : C}$$

T-LET

$$\frac{\Delta; \Gamma \vdash M : A!E \quad \Delta; \Gamma, x : A \vdash N : B!E}{\Delta; \Gamma \vdash \mathbf{let} \ x \leftarrow M \mathbf{in} \ N : B!E}$$

T-RETURN

Effect and handler typing rules

T-Do

$$\frac{\Delta; \Gamma \vdash V : A \quad E = \{\ell : A \rightarrow B; R\}}{\Delta; \Gamma \vdash (\mathbf{do} \; \ell \; V)^E : B!E}$$

T-HANDLE

$$\frac{\Delta; \Gamma \vdash M : C \quad \Delta; \Gamma \vdash H : C \Rightarrow D}{\Delta; \Gamma \vdash \mathbf{handle} \; M \; \mathbf{with} \; H : D}$$

T-HANDLER

$$\frac{\begin{array}{c} C = A! \{(\ell_i : A_i \rightarrow B_i)_i; R\} \quad D = B! \{(\ell_i : P_i)_i; R\} \\ H = \{\mathbf{return} \; x \mapsto M\} \uplus \{\ell_i \; y \; k \mapsto N_i\}_i \\ [\Delta; \Gamma, y : A_i, k : B_i \rightarrow D \vdash N_i : D]_i \quad \Delta; \Gamma, x : A \vdash M : D \end{array}}{\Delta; \Gamma \vdash H : C \Rightarrow D}$$

Operational semantics for effect handlers

S-HANDLE-RET

handle (return V) with $H \rightsquigarrow M[V/x]$,
where $\{\mathbf{return} \ x \mapsto M\} \in H$

S-HANDLE-OP

handle $\mathcal{E}[\mathbf{do} \ \ell \ V]$ with $H \rightsquigarrow$
 $M[V/x, \lambda y. \mathbf{handle} \ \mathcal{E}[\mathbf{return} \ y] \ \mathbf{with} \ H/k]$,
where $\ell \notin BL(\mathcal{E})$ and $\{\ell \ x \mapsto M\} \in H$

Evaluation contexts

$\mathcal{E} ::= [] \mid \mathbf{let} \ x \leftarrow \mathcal{E} \ \mathbf{in} \ N \mid \mathbf{handle} \ \mathcal{E} \ \mathbf{with} \ H$

Abstract machine syntax

Configurations	$\mathcal{C} ::= \langle M \mid \gamma \mid \kappa \rangle$ $\langle M \mid \gamma \mid \kappa \mid \kappa' \rangle_{\text{op}}$
Value environments	$\gamma ::= \emptyset \mid \gamma[x \mapsto v]$
Values	$v, w ::= \lambda^\gamma x^A.M \mid \Lambda^\gamma \alpha^K.M$ $\langle \rangle \mid \langle \ell = v; w \rangle \mid (\ell v)^R \mid \kappa^A$
Continuations	$\kappa ::= [] \mid \delta :: \kappa$
Continuation frames	$\delta ::= (\sigma, \chi)$
Pure continuations	$\sigma ::= [] \mid \phi :: \sigma$
Pure continuation frames	$\phi ::= (\gamma, x, N)$
Handlers	$\chi ::= (\gamma, H)$

Abstract machine semantics

$\langle V \ W \mid \gamma \mid \kappa \rangle \longrightarrow \langle M \mid \gamma'[x \mapsto \llbracket W \rrbracket \gamma] \mid \kappa \rangle,$	if $\llbracket V \rrbracket \gamma = \lambda^{\gamma'} x. M$
$\langle V \ W \mid \gamma \mid \kappa \rangle \longrightarrow \langle \mathbf{return} \ W \mid \gamma \mid \kappa' \uplus \kappa \rangle,$	if $\llbracket V \rrbracket \gamma = \kappa'$
$\langle V \ A \mid \gamma \mid \kappa \rangle \longrightarrow \langle M[A/\alpha] \mid \gamma' \mid \kappa \rangle,$	if $\llbracket V \rrbracket \gamma = \Lambda^{\gamma'} \alpha. M$
$\langle \mathbf{let} \ \langle \ell = x; y \rangle \leftarrow V \ \mathbf{in} \ N \mid \gamma \mid \kappa \rangle \longrightarrow \langle N \mid \gamma[x \mapsto v, y \mapsto w] \mid \kappa \rangle,$	if $\llbracket V \rrbracket \gamma = \langle \ell = v; w \rangle$
$\langle \mathbf{case} \ V \ \{ \ell x \mapsto M; y \mapsto N \} \mid \gamma \mid \kappa \rangle \longrightarrow \begin{cases} \langle M \mid \gamma[x \mapsto v] \mid \kappa \rangle, \\ \langle N \mid \gamma[y \mapsto \ell' v] \mid \kappa \rangle, \end{cases}$	if $\llbracket V \rrbracket \gamma = \ell' v$ and $\ell \neq \ell'$
$\langle \mathbf{let} \ x \leftarrow M \ \mathbf{in} \ N \mid \gamma \mid (\sigma, \chi) :: \kappa \rangle \longrightarrow \langle M \mid \gamma \mid ((\gamma, x, N) :: \sigma, \chi) :: \kappa \rangle$	
$\langle \mathbf{handle} \ M \ \mathbf{with} \ H \mid \gamma \mid \kappa \rangle \longrightarrow \langle M \mid \gamma \mid ([], (\gamma, H)) :: \kappa \rangle$	
$\langle \mathbf{return} \ V \mid \gamma \mid ((\gamma', x, N) :: \sigma, \chi) :: \kappa \rangle \longrightarrow \langle N \mid \gamma'[x \mapsto \llbracket V \rrbracket \gamma] \mid (\sigma, \chi) :: \kappa \rangle$	
$\langle \mathbf{return} \ V \mid \gamma \mid ([], (\gamma', H)) :: \kappa \rangle \longrightarrow \langle M \mid \gamma'[x \mapsto \llbracket V \rrbracket \gamma] \mid \kappa \rangle,$	if $H(\mathbf{return}) = \{\mathbf{return} \ x \mapsto M\}$
$\langle \mathbf{return} \ V \mid \gamma \mid [] \rangle \longrightarrow \llbracket V \rrbracket \gamma$	
$\langle (\mathbf{do} \ \ell \ V)^E \mid \gamma \mid \kappa \rangle \longrightarrow \langle (\mathbf{do} \ \ell \ V)^E \mid \gamma \mid \kappa \mid [] \rangle_{\text{op}}$	
$\langle (\mathbf{do} \ \ell \ V)^E \mid \gamma \mid \delta :: \kappa \mid \kappa' \rangle_{\text{op}} \longrightarrow \langle M \mid \gamma'[x \mapsto \llbracket V \rrbracket \gamma, k \mapsto (\kappa' \uplus [\delta])^B] \mid \kappa \rangle,$	if $\ell : A \rightarrow B \in E$ and $\delta(\ell) = \{\ell \ x \ k \mapsto M\}$
$\langle (\mathbf{do} \ \ell \ V)^E \mid \gamma \mid \delta :: \kappa \mid \kappa' \rangle_{\text{op}} \longrightarrow \langle (\mathbf{do} \ \ell \ V)^E \mid \gamma \mid \kappa \mid \kappa' \uplus [\delta] \rangle_{\text{op}},$	if $\delta(\ell) = \emptyset$

Value interpretation

$$\begin{aligned}\llbracket x \rrbracket \gamma &= \gamma(x) \\ \llbracket \lambda x^A.M \rrbracket \gamma &= \lambda^\gamma x^A.M \\ \llbracket \Lambda \alpha^K.M \rrbracket \gamma &= \Lambda^\gamma \alpha^K.M \\ \llbracket \langle \rangle \rrbracket \gamma &= \langle \rangle \\ \llbracket \langle \ell = V; W \rangle \rrbracket \gamma &= \langle \ell = \llbracket V \rrbracket \gamma; \llbracket W \rrbracket \gamma \rangle \\ \llbracket (\ell V)^R \rrbracket \gamma &= (\ell \llbracket V \rrbracket \gamma)^R\end{aligned}$$

Mapping configurations to terms

Configurations

$$\begin{aligned}\langle\langle M \mid \gamma \mid \kappa\rangle\rangle &= \langle\kappa\rangle(M, \gamma) \\ \langle\langle M \mid \gamma \mid \kappa \mid \kappa'\rangle_{\text{op}}\rangle &= \langle\kappa' + \kappa\rangle(M, \gamma) = \langle\kappa'\rangle(\langle\kappa\rangle(M, \gamma), \emptyset)\end{aligned}$$

Continuations

$$\begin{aligned}\langle[]\rangle(M, \gamma) &= \langle M \rangle \gamma \\ \langle((\gamma', x, N) :: \sigma, \chi) :: \kappa\rangle(M, \gamma) &= \langle(\sigma, \chi) :: \kappa\rangle(\mathbf{let} \ x \leftarrow M \ \mathbf{in} \ \langle N \rangle(\gamma' \setminus \{x\}), \gamma) \\ \langle(\[], (\gamma', H)) :: \kappa\rangle(M, \gamma) &= \langle\kappa\rangle(\mathbf{handle} \ M \ \mathbf{with} \ \langle H \rangle \gamma', \gamma)\end{aligned}$$

Computation terms

$$\begin{aligned}\langle V \ W \rangle \gamma &= \langle V \rangle \gamma \langle W \rangle \gamma \\ \langle V \ A \rangle \gamma &= \langle V \rangle \gamma \ A\end{aligned}$$

$$\begin{aligned}\langle \mathbf{let} \ \langle \ell = x; y \rangle \leftarrow V \ \mathbf{in} \ N \rangle \gamma &= \mathbf{let} \ \langle \ell = x; y \rangle \leftarrow \langle V \rangle \gamma \ \mathbf{in} \ \langle N \rangle(\gamma \setminus \{x, y\}) \\ \langle \mathbf{case} \ V \ \{ \ell \ x \mapsto M; y \mapsto N \} \rangle \gamma &= \mathbf{case} \ \langle V \rangle \gamma \ \{ \ell \ x \mapsto \langle M \rangle(\gamma \setminus \{x\}); y \mapsto \langle N \rangle(\gamma \setminus \{y\}) \} \\ \langle \mathbf{return} \ V \rangle \gamma &= \mathbf{return} \ \langle V \rangle \gamma \\ \langle \mathbf{let} \ x \leftarrow M \ \mathbf{in} \ N \rangle \gamma &= \mathbf{let} \ x \leftarrow \langle M \rangle \gamma \ \mathbf{in} \ \langle N \rangle(\gamma \setminus \{x\}) \\ \langle \mathbf{do} \ \ell \ V \rangle \gamma &= \mathbf{do} \ \ell \ \langle V \rangle \gamma \\ \langle \mathbf{handle} \ M \ \mathbf{with} \ H \rangle \gamma &= \mathbf{handle} \ \langle M \rangle \gamma \ \mathbf{with} \ \langle H \rangle \gamma\end{aligned}$$

Handler definitions

$$\begin{aligned}\{\{\mathbf{return} \ x \mapsto M\}\} \gamma &= \{\mathbf{return} \ x \mapsto \langle M \rangle(\gamma \setminus \{x\})\} \\ \{\{\ell \ x \ k \mapsto M\} \uplus H\} \gamma &= \{\ell \ x \ k \mapsto \langle M \rangle(\gamma \setminus \{x, k\})\} \uplus \langle H \rangle \gamma\end{aligned}$$

Mapping configurations to terms (continued)

Value terms and values

$$\langle\!\langle x \rangle\!\rangle \gamma = \langle\!\langle v \rangle\!\rangle, \quad \text{if } \gamma(x) = v \\ \langle\!\langle x \rangle\!\rangle \gamma = x, \quad \text{if } x \notin \text{dom}(\gamma)$$

$$\langle\!\langle \lambda x^A.M \rangle\!\rangle \gamma = \lambda x^A. \langle\!\langle M \rangle\!\rangle (\gamma \setminus \{x\}) \\ \langle\!\langle \Lambda \alpha^K.M \rangle\!\rangle \gamma = \Lambda \alpha^K. \langle\!\langle M \rangle\!\rangle \gamma \\ \langle\!\langle \langle \rangle \rangle \gamma = \langle \rangle$$

$$\langle\!\langle \langle \ell = V; W \rangle \rangle \gamma = \langle \ell = \langle\!\langle V \rangle\!\rangle \gamma; \langle\!\langle W \rangle\!\rangle \gamma \rangle \\ \langle\!\langle (\ell \ V)^R \rangle\!\rangle \gamma = (\ell \ \langle\!\langle V \rangle\!\rangle \gamma)^R$$

$$\langle\!\langle \lambda^\gamma x^A.M \rangle\!\rangle = \lambda x^A. \langle\!\langle M \rangle\!\rangle (\gamma \setminus \{x\}) \\ \langle\!\langle \Lambda^\gamma \alpha^K.M \rangle\!\rangle = \Lambda \alpha^K. \langle\!\langle M \rangle\!\rangle \gamma \\ \langle\!\langle \langle \rangle \rangle \rangle = \langle \rangle \\ \langle\!\langle \langle \ell = v; w \rangle \rangle \rangle = \langle \ell = \langle\!\langle v \rangle\!\rangle; \langle\!\langle w \rangle\!\rangle \rangle \\ \langle\!\langle (\ell \ v)^R \rangle\!\rangle = (\ell \ \langle\!\langle v \rangle\!\rangle)^R \\ \langle\!\langle \kappa^A \rangle\!\rangle = \lambda x^A. \langle\!\langle \kappa \rangle\!\rangle (\mathbf{return} \ x, \emptyset)$$

Theorems

Definition

Computation term N is normal with respect to effect E , if N is either of the form **return** V , or $\mathcal{E}[\mathbf{do} \ell W]$, where $\ell \in E$ and $\ell \notin BL(\mathcal{E})$.

Theorem (Type Soundness)

If $\vdash M : A!E$, then there exists $\vdash N : A!E$, such that $M \rightsquigarrow^+ N \not\rightsquigarrow$, and N is normal with respect to effect E .

Definition

$$\implies = \longrightarrow_a^* \longrightarrow_\beta$$

Theorem (Simulation)

If $M \rightsquigarrow N$, then for any \mathcal{C} , such that $\langle\!\langle \mathcal{C} \rangle\!\rangle = M$, there exists \mathcal{C}' , such that $\mathcal{C} \implies \mathcal{C}'$ and $\langle\!\langle \mathcal{C}' \rangle\!\rangle = N$.

- ▶ bidirectional type system
- ▶ shadowing instead of presence information
- ▶ shallow handlers
- ▶ everything is a handler (or rather *multihandler*)
 - ▶ handlers (multihandlers) generalise functions
 - ▶ “call-by-handling” generalises call-by-value
- ▶ effect polymorphism with a single invisible effect variable

Related abstractions

- ▶ monad transformers
- ▶ free monads
 - ▶ deep handler: fold over free monad
 - ▶ shallow handler: case-split over free monad
- ▶ containers
- ▶ monadic reflection
- ▶ exceptional syntax
- ▶ delimited continuations
 - ▶ Bauer:

“effects + handlers” : “delimited continuations”

=

“while” : “goto”

- ▶ deep handlers: shift0/reset0
- ▶ shallow handlers: control0/prompt0
- ▶ modules, type classes, dynamic binding

Some related work

Algebraic effects [Plotkin and Power, 2001]

Effect handlers [Pretnar and Plotkin, 2009]

Other effect handler languages and libraries:

- ▶ Eff [Bauer and Pretnar, 2012]
- ▶ Haskell, SML, OCaml, Racket, Scheme effect libraries [Kammar, Lindley, Oury, 2013]
- ▶ Haskell extensible effects [Kiselyov, Sabry, Swords, 2013]
- ▶ Idris effects library [Brady, 2013]
- ▶ Experimental OCaml development branch [Dolan, Silvaramakrishnan, White, Yallop, Madhavapeddy, 2015]
- ▶ Prolog library [Schrijvers, Wu, Desouter, Demoen, 2014]
- ▶ Shonky [McBride, 2016]

Other stuff:

- ▶ Extensible denotational language specifications [Cartwright and Felleisen, 1994]
- ▶ Data types a la carte [Swierstra, 2008]
- ▶ Kleisli arrows of outrageous fortune [McBride, 2011]
- ▶ Scoped effect handlers [Wu, Schrijvers, Hinze, 2014]
- ▶ Fusion for free [Wu and Schrijvers, 2015]