

Which simple types have a unique inhabitant?

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- in a given type system (STLC with atoms, products and sums)
- modulo some program equivalence ( $\beta\eta$ )

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Motivation: a principal way to study **code inference**.

We can infer the instance declaration for the exception monad  $A \mapsto A + E$ :

$$X + E \rightarrow (X \rightarrow Y + E) \rightarrow Y + E$$

## STLC with sums

$$\frac{\Delta, x : A \vdash t : B}{\Delta \vdash \lambda(x) t : A \rightarrow B}$$

$$\frac{\Delta \vdash t : A \rightarrow B \quad \Delta \vdash u : A}{\Delta \vdash t u : B}$$

$$\frac{\Delta \vdash t : A \quad \Delta \vdash u : B}{\Delta \vdash (t, u) : A * B}$$

$$\frac{\Delta \vdash t : A_1 * A_2}{\Delta \vdash \pi_i t : A_i}$$

$$\Delta, x : A \vdash x : A$$

$$\frac{\Delta \vdash t : A_i}{\Delta \vdash \sigma_i t : A_1 + A_2}$$

$$\frac{\Delta \vdash t : A_1 + A_2 \quad \Delta, x_1 : A_1 \vdash u_1 : C \quad \Delta, x_2 : A_2 \vdash u_2 : C}{\Delta \vdash \delta(t, x_1.u_1, x_2.u_2) : C}$$

## $\beta\eta$ -equivalence

$$(\lambda(x) t) u \rightarrow_{\beta} t[u/x] \qquad (t : A \rightarrow B) =_{\eta} \lambda(x) t x$$

$$\pi_i (t_1, t_2) \rightarrow_{\beta} t_i \qquad (t : A * B) =_{\eta} (\pi_1 t, \pi_2 t)$$

$$\delta(\sigma_i t, x_1.u_1, x_2.u_2) \rightarrow_{\beta} u_i[t/x_i]$$

$$\forall C[\square], \quad C[t : A + B] =_{\eta} \delta(t, x.C[\sigma_1 x], x.C[\sigma_2 x])$$

Equivalence algorithm decidable — since 1995.

You said  $\beta$ -short  $\eta$ -long normal forms?

In presence of negative connectives only (or positive only), **focused** proof search enumerates distinct normal forms.

This fails when sums (positives) are mixed with arrows (negatives).

## The obvious idea...

Enumerate all derivations in a reasonable (focused) system.

Remove duplicates using the equivalence algorithm.

Stop if two proofs are found.

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We need a more canonical proof search process.

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We consider search processes of the form “enumerating the derivations of this restricted system of inference rules”.

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- Termination — of failure

Focused proof search: computationally complete, not canonical.

Focused proof search quotiented by equivalence: complete, canonical, non-terminating.

## Our approach

We promised an algorithm that decides uniqueness of inhabitation.

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We distinguish **specification** and **implementation**.

Specification: a novel focused logic that is **computationally complete** and **canonical**.

Implementation: a restriction of this logic that is **unicity complete** and **terminating**.

## The core idea

Some proof rules are **invertible** and some are not – they lead to dead ends.

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(Focusing separates invertible and non-invertible steps.)

To avoid **all** dead ends, we will **saturate** by deducing everything we can (on the left) before any risky move on the right.

“Cut the positives as soon as possible.”

# Polarized simply-typed lambda-calculus, with non-biased atoms

$A, B, C, D ::=$		types
	$X, Y, Z$	atoms
	$P, Q$	positive types
	$N, M$	negative types

$P, Q$	$::= A + B$	strict positive
$N, M$	$::= A \rightarrow B \mid A * B$	strict negative

$P_{\text{at}}, Q_{\text{at}}$	$::= P, Q \mid X, Y, Z$	positive or atomic
$N_{\text{at}}, M_{\text{at}}$	$::= N, M \mid X, Y, Z$	negative or atomic

$\Gamma$	$::= \text{varmap}(N_{\text{at}})$	negative or atomic context
$\Delta$	$::= \text{varmap}(A)$	general context

# Focused natural deduction for intuitionistic logic

INV-SUM

$$\frac{\Gamma; \Delta, x : A \vdash_{\text{inv}} t : C \quad \Gamma; \Delta, x : B \vdash_{\text{inv}} u : C}{\Gamma; \Delta, x : A + B \vdash_{\text{inv}} \delta(x, x.t, x.u) : C}$$

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$$\frac{\Gamma; \Delta, x : A \vdash_{\text{inv}} t : B}{\Gamma; \Delta \vdash_{\text{inv}} \lambda(x) t : A \rightarrow B}$$

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$$\frac{\Gamma; \Delta \vdash_{\text{inv}} t : A \quad \Gamma; \Delta \vdash_{\text{inv}} u : B}{\Gamma; \Delta \vdash_{\text{inv}} (t, u) : A * B}$$

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$$\frac{\Gamma, \Gamma' \vdash_{\text{foc}} t : P_{\text{at}}}{\Gamma; \Gamma' \vdash_{\text{inv}} t : P_{\text{at}}}$$

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FOC-ELIM

$$\frac{\Gamma \vdash n \Downarrow P \quad \Gamma; x : P \vdash_{\text{inv}} t : Q_{\text{at}}}{\Gamma \vdash_{\text{foc}} \text{let } x = n \text{ in } t : Q_{\text{at}}}$$

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INTRO-SUM

$$\frac{\Gamma \vdash t \Uparrow A_i}{\Gamma \vdash \sigma_i t \Uparrow A_1 + A_2}$$

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$$\frac{\Gamma; \emptyset \vdash_{\text{inv}} t : N_{\text{at}}}{\Gamma \vdash t \Uparrow N_{\text{at}}}$$

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ELIM-ARR

$$\frac{\Gamma \vdash n \Downarrow A \rightarrow B \quad \Gamma \vdash u \Uparrow A}{\Gamma \vdash n u \Downarrow B}$$

# Saturating focused natural deduction

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saturation

$$X, (X \rightarrow Y + Y) \vdash Y$$

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canonicity

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finite derivations

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$$\frac{(x : N_{\text{at}}) \in \Gamma}{\Gamma \vdash x \Downarrow N_{\text{at}}}$$
 <sup>12</sup>

ELIM-ARR

$$\frac{\Gamma \vdash n \Downarrow A \rightarrow B \quad \Gamma \vdash u \Uparrow A}{\Gamma \vdash n u \Downarrow B}$$

# Saturating focused natural deduction

SINV-SUM

$$\frac{\Gamma; \Delta, x : A \vdash_{\text{sinv}} t : C \quad \Gamma; \Delta, x : B \vdash_{\text{sinv}} u : C}{\Gamma; \Delta, x : A + B \vdash_{\text{sinv}} \delta(x, x.t, x.u) : C}$$

SINV-PAIR

$$\frac{\Gamma; \Delta \vdash_{\text{sinv}} t : A \quad \Gamma; \Delta \vdash_{\text{sinv}} u : B}{\Gamma; \Delta \vdash_{\text{sinv}} (t, u) : A * B}$$

SINV-ARR

$$\frac{\Gamma; \Delta, x : A \vdash_{\text{sinv}} t : B}{\Gamma; \Delta \vdash_{\text{sinv}} \lambda(x) t : A \rightarrow B}$$

SINV-END

$$\frac{\Gamma; \Gamma' \vdash_{\text{sat}} t : P_{\text{at}}}{\Gamma; \Gamma' \vdash_{\text{sinv}} t : P_{\text{at}}}$$

$$(\bar{n}, \bar{P}) \subseteq \{(n, P) \mid (\Gamma, \Gamma' \vdash n \Downarrow P) \wedge n \text{ uses } \Gamma'\}$$

$$\Gamma, \Gamma'; \bar{x} : \bar{P} \vdash_{\text{sinv}} t : Q_{\text{at}} \quad \forall x \in \bar{x}, t \text{ uses } x$$

$$\frac{}{\Gamma; \Gamma' \vdash_{\text{sat}} \text{let } \bar{x} = \bar{n} \text{ in } t : Q_{\text{at}}}$$

SAT-INTRO

$$\frac{\Gamma \vdash t \Uparrow P}{\Gamma; \emptyset \vdash_{\text{sat}} t : P}$$

SAT-ATOM

$$\frac{\Gamma \vdash n \Downarrow X}{\Gamma; \emptyset \vdash_{\text{sat}} n : X}$$

INTRO-SUM

$$\frac{\Gamma \vdash t \Uparrow A_i}{\Gamma \vdash \sigma_i t \Uparrow A_1 + A_2}$$

INTRO-END

$$\frac{\Gamma; \emptyset \vdash_{\text{sinv}} t : N_{\text{at}}}{\Gamma \vdash t \Uparrow N_{\text{at}}}$$

ELIM-PAIR

$$\frac{\Gamma \vdash n \Downarrow A_1 * A_2}{\Gamma \vdash \pi_i n \Downarrow A_i}$$

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Breaks computational completeness

## Termination 1: bounded multisets

There exists a  $M_2 \in \mathbb{N}$  such that, by keeping at most  $M_2$  variables of each type/formula in  $\Gamma$ , then we can find at least 2 distinct proofs of  $\Gamma \vdash A$  if they exist.

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In fact  $M_2 := 2$  suffices.

## Termination 2: recurring at most twice

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...

Computational completeness is lost, but  
Unicity completeness regained.

# Algorithm

Our algorithm searches for all saturated proofs under these two search-space restriction.

Optimization 1: redundancy (elim and intro).

Optimization 2: monotonicity.

# (The logic again)

SINV-SUM

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SAT-INTRO

$$\frac{\Gamma \vdash t \uparrow P}{\Gamma; \emptyset \vdash_{\text{sat}} t : P}$$

SAT-ATOM

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INTRO-SUM

$$\frac{\Gamma \vdash t \uparrow A_i}{\Gamma \vdash \sigma_i t \uparrow A_1 + A_2}$$

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Future work: extend to polymorphism and dependent types.

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Thanks. Any question?

Paper draft:

[gallium.inria.fr/~scherer/drafts/unique\\_stlc\\_sums.pdf](http://gallium.inria.fr/~scherer/drafts/unique_stlc_sums.pdf)